23. Preservation and adjunction

Diagram $D\colon J \to \mathbf{A}$, adjoint functors F, U:

$$\begin{tikzcd} J \arrow[d, "D"'] & \\ \mathbf{A} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathbf{X} \arrow[l, bend right, "F"'] \end{tikzcd}$$

Suppose limit $\left[\varprojlim D \xrightarrow{\pi_j} D_j\right]$ exists:

$$\begin{tikzcd} A \arrow[rr, "\kappa_j"] \arrow[dr, "r=\varprojlim \kappa"'] & & D_j \\ & \varprojlim D \arrow[ur, "\pi_j"'] & \end{tikzcd}$$

Thus $\mathbf{A}^J(\Delta A, D) \cong \mathbf{A}(A, \varprojlim D)$.
$\qquad \kappa_j = \pi_j \circ r \mapsto \qquad r$

Theorem: Right adjoints preserve limits.

Proof. UD has limit $U \varprojlim D$:
$\mathbf{X}^J(\Delta X, UD) \cong \mathbf{A}^J\big(\Delta(FX), D\big) \cong \mathbf{A}(FX, \varprojlim D) \cong \mathbf{X}(X, U \varprojlim D)\,.$ $\square$

Corollary: Left adjoints preserve colimits.

Example: In $\mathcal{L}(FX, V) \cong \mathbf{Set}(X, UV)$,
have $U(V_1 \oplus V_2) = V_1 \times V_2$ and $F(X_1 + X_2) = FX_1 \oplus FX_2$.

Example: Multiplicity of the **Euler φ-function** or **totient function**

$$\varphi(n) = |\{r \mid 1 \le r \le n \text{ and } \gcd(r,n) = 1\}| = \big|(\mathbb{Z}/_n, \times, 1)^*\big|.$$

Recall group of Units functor $U\colon \mathbf{Mon} \to \mathbf{Grp}; (M, \cdot, 1) \mapsto (M^*, \cdot, ^{-1}, 1)$
is right adjoint to Forgetful $F\colon \mathbf{Grp} \to \mathbf{Mon}; (G, \cdot, ^{-1}, 1) \mapsto (G, \cdot, 1)$.

For coprime m, n, have $(\mathbb{Z}/_{mn}, \times, 1) \cong (\mathbb{Z}/_m, \times, 1) \times (\mathbb{Z}/_n, \times, 1)$.

Then $\varphi(mn) = \big|(\mathbb{Z}/_{mn}, \times, 1)^*\big| = \Big|\big[(\mathbb{Z}/_m, \times, 1) \times (\mathbb{Z}/_n, \times, 1)\big]^*\Big|$
$= \big|(\mathbb{Z}/_m, \times, 1)^* \times (\mathbb{Z}/_n, \times, 1)^*\big| = \big|(\mathbb{Z}/_m, \times, 1)^*\big| \times \big|(\mathbb{Z}/_n, \times, 1)^*\big| = \varphi(m)\varphi(n).$

Example: Equivalence $\begin{tikzcd} \mathbf{A} \arrow[r, bend right, "U"'] & \mathbf{X} \arrow[l, bend right, "F"'] \end{tikzcd}$

implies $\begin{tikzcd} \mathbf{A} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathbf{X} \arrow[l, bend right, "F"'] \end{tikzcd}$ and $\begin{tikzcd} \mathbf{A} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\top"] & \mathbf{X} \arrow[l, bend right, "F"'] \end{tikzcd}$,

so F and U preserve limits and colimits.