22. Preservation, reflection and creation

Diagram $D\colon J \to \mathbf{A}$, functor $G\colon \mathbf{A} \to \mathbf{B}$.

$$\begin{array}{ccc} J & & \\ {\scriptstyle D}\downarrow & & \\ \mathbf{A} & \xrightarrow[G]{} & \mathbf{B} \end{array}$$

G **preserves** J-limits if it "pushes limits forward":
Diagram $D\colon J \to \mathbf{A}$ has a limit $\left[\varprojlim D \xrightarrow{\pi_j} D_j\right]$
implies $GD\colon J \to \mathbf{B}$ has a limit $\left[G(\varprojlim D) \xrightarrow{G\pi_j} GD_j\right]$.

G **reflects** J-limits if it "pulls limits back":
Diagram $GD\colon J \to \mathbf{B}$ has a limit of the **image form** $\left[GL \xrightarrow{G\pi_j} GD_j\right]$
implies $D\colon J \to \mathbf{A}$ already had a limit $\left[\varprojlim D = L \xrightarrow{\pi_j} D_j\right]$.

G **creates** J-limits if it both preserves and reflects,
and if $\varprojlim GD$ exists, then it exists in the image form.

Corresponding definitions for colimits.

Example: $U\colon \mathbf{Grp} \to \mathbf{Set}$ preserves, reflects limits, directed colimits.
[Consider "pointwise" structure on the underlying sets.]
Doesn't preserve or reflect general colimits.

Example: $U\colon \mathbf{Top} \to \mathbf{Set}$ preserves, but doesn't reflect, limits:

$$\begin{array}{ccc} E & \xrightarrow{\pi_Y} & Y \\ {\scriptstyle \pi_X}\downarrow & \text{p-b} & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & B \end{array} \qquad \text{in } \mathbf{Top}$$

means $E = \{(x,y) \in X \times Y \mid f(x) = g(y)\}$ has the subspace topology.

Example:
Full and faithful $K\colon \mathbf{Ab} \hookrightarrow \mathbf{Grp}$ preserves limits, but not colimits.

Theorem: Full and faithful $G\colon \mathbf{A} \to \mathbf{B}$ reflects limits and colimits.

Example: In **Ab**, coproduct $C_2 + C_3$ or $C_2 \oplus C_3$ is C_6.
In **Grp**, coproduct $C_2 + C_3$ or $C_2 * C_3$ is the **modular group** $\mathsf{PSL}_2(\mathbb{Z})$.
Doesn't violate $K\colon \mathbf{Ab} \hookrightarrow \mathbf{Grp}$ reflecting colimits: $\mathsf{PSL}_2(\mathbb{Z}) \neq KC_6$.