21. Typical equivalences

• **Skeleton** S of C: unique representative for each isomorphism class.
Like poset $(P, \leq)$ induced in preorder $(Q, \leq)$,
essentially surjective $K\colon S \hookrightarrow C$ has reflection $L\colon C \to S$.

Ex: $\big\{\{i \in \mathbb{N} \mid i < n\} \,\big|\, n \in \mathbb{N}\big\}$ as object set of skeleton of **FinSet**.

• **Morita equivalence:** Ring R, ring R_n^n of $n \times n$-matrices over R.

$$\begin{tikzcd} \mathbf{Mod}_R \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathbf{Mod}_{R_n^n} \arrow[l, bend right, "F"'] \end{tikzcd} \quad \text{with } U\colon M \to \overbrace{M \oplus \cdots \oplus M}^{n},$$

$$F\colon [R_n^n \to \mathrm{End}(N)] \mapsto [R \to R_n^n \to \mathrm{End}(N)].$$

Concrete category:
Category of sets with structure (algebraic, topological,. . .)
and structure-preserving functions (homomorphisms, continuous,. . .).

• **Duality:** Equivalence $\begin{tikzcd} \mathbf{A} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathbf{X}^{\mathsf{op}} \arrow[l, bend right, "F"'] \end{tikzcd}$ of concrete categories.

Dualizing object: Set T with structure $T \in \mathbf{A}$ or $T \in \mathbf{X}$,
where: $\forall\ A \in \mathbf{A}_0\,,\ \mathbf{A}(A, T) \leq \mathbf{Set}(A, T) = T^A \in \mathbf{X}$
and: $\forall\ X \in \mathbf{X}_0\,,\ \mathbf{X}(X, T) \leq \mathbf{Set}(X, T) = T^X \in \mathbf{A}$.

Then $U = \mathbf{A}(\_\_, T)$ and $F = \mathbf{X}(\_\_, T)$.

Example: Category $\mathcal{L}_{\mathsf{fin}}$ of fin.-dim. vector spaces over a field K.
Then $\mathbf{A} = \mathbf{X} = \mathcal{L}_{\mathsf{fin}}$, $T = K$ and $\varepsilon_V^{-1}\colon V \to V^{**}; v \mapsto [f \mapsto f(v)]$.

Example: Fourier transforms, **Pontryagin duality**.

Then $\mathbf{A} = \mathbf{Ab}$, $\mathbf{X} = \mathbf{CAb}$ (compact abelian groups),
and $T = (\mathbb{R}/\mathbb{Z}, +, 0)$ "1-dimensional torus" or $(S^1, \cdot, 1)$ "circle group".
$\widehat{A} := UA = \mathbf{Ab}(A, T)$, the group of **characters** $\chi\colon A \to T$.

$$\varepsilon_A^{-1}\colon A \to FUA; a \mapsto [\chi \mapsto \chi(a)].$$

Example: Category $\mathbf{A}$ of finite Boolean algebras, $\mathbf{X} = \mathbf{FinSet}$,
dualizing object $T = \mathbf{2} := \{0, 1\}$, so power set FX (char. fns.).

$$\eta_X\colon X \to UFX; x \mapsto [\chi \mapsto \chi(x)].$$

Note: Can extend from a category $\mathbf{A}_{\mathsf{f.g.}}$ of finitely generated algebras to a category $\mathbf{A}$ of all algebras: treat as colimits of f.g. algebras, which will dualize to limits of $\mathbf{X}_{\mathsf{f.g.}}$-objects.