20. Category equivalence

Equivalence: Full, faithful, essentially surjective functor $F\colon \mathbf{X} \to \mathbf{A}$.

Recall **essentially surjective:** $\forall\ A \in \mathbf{A}_0\,,\ \exists\ X \in \mathbf{X}\,.\ \varepsilon_A\colon FX \cong A\,.$

Preorder: Set $(Q, \leq)$ with reflexive transitive relation $\leq$ on set Q,
or a small category with $\forall\ x, y \in Q\,,\ |Q(x,y)| \leq 1$.

Define α on Q by $x\ \alpha\ y \ \Leftrightarrow\ x \leq y$ and $y \leq x$, an equivalence relation.
Set P of equivalence class representatives: $\forall\ q \in Q\,,\ \exists\ p \in P\,.\ p \cong q\,.$
Inclusion functor $F\colon (P, \leq) \hookrightarrow (Q, \leq)$ is an equivalence.
"Election" functor $U\colon (Q, \leq) \to (P, \leq)$ chooses representatives.
Then $\forall\ q \in Q\,,\ \varepsilon_q\colon FUq \cong q$, isomorphic counit of an adjunction.
Note $(P, \leq)$ is a poset — antireflexive!

Adjoint equivalence: $\begin{tikzcd} \mathbf{A} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathbf{X} \arrow[l, bend right, "F"'] \end{tikzcd}$ with unit, counit iso.

Equivalence: $\begin{tikzcd} \mathbf{A} \arrow[r, bend right, "U"'] & \mathbf{X} \arrow[l, bend right, "F"'] \end{tikzcd}$ with $1_{\mathbf{X}} \to UF$, $\quad FU \to 1_{\mathbf{A}}$ iso.

Theorem: Functor $F\colon \mathbf{X} \to \mathbf{A}$. TFAE: (a) F is an equivalence;
(b) F is part of an adjoint equivalence of categories;
(c) F is part of an equivalence of categories.

(a)$\Rightarrow$(b): $\forall\ A \in \mathbf{A}_0\,,\ \exists\ UA \in \mathbf{X}\,.\ \varepsilon_A\colon FUA \cong A\,.$ Full, faithful $F \Rightarrow$
$\forall\ f \in \mathbf{A}(FX, A)\,,\ \exists_!\ \varphi_{X,A}f \in \mathbf{X}(X, UA)\,.\ F\varphi_{X,A}f = \varepsilon_A^{-1} \circ f,\ \ldots$
[Complete the adjunction, dual to the construction for linear algebra.]

(c)$\Rightarrow$(a): Need F full and faithful.

F faithful: $\begin{tikzcd} X_1 \arrow[r, "\eta_{X_1}"] \arrow[d, "g"'] & UFX_1 \arrow[d, "UFg"] \\ X_2 \arrow[r, "\eta_{X_2}"'] & UFX_2 \end{tikzcd}$ and U faithful: $\begin{tikzcd} FUA_1 \arrow[r, "\varepsilon_{A_1}"] \arrow[d, "k"'] & A_1 \arrow[d, "FUk"] \\ FUA_2 \arrow[r, "\varepsilon_{A_2}"] & A_2 \end{tikzcd}$

F full: For $h \in \mathbf{A}(FX_1, FX_2)$, want $h = Ff$ for $f \in \mathbf{X}(X_1, X_2)$.

Have $\begin{tikzcd} X_1 \arrow[r, "\eta_{X_1}"] \arrow[d, "f"'] & UFX_1 \arrow[d, "Uh"] \\ X_2 \arrow[r, "\eta_{X_2}"'] & UFX_2 \end{tikzcd}$ for $f = \eta_{X_2}^{-1} \circ Uh \circ \eta_{X_1}$ and $\begin{tikzcd} X_1 \arrow[r, "\eta_{X_1}"] \arrow[d, "f"'] & UFX_1 \arrow[d, "UFf"] \\ X_2 \arrow[r, "\eta_{X_2}"'] & UFX_2 \end{tikzcd}$
so $Uh = UFf$. Then U faithful gives $h = Ff$. $\quad\square$

Corollary: Essentially surjective $K\colon \mathbf{A} \hookrightarrow \mathbf{B}$ gives a reflection.