19. Reflective subcategories and counit properties

Reflective subcategory $\mathbf{A}$ of $\mathbf{B}$ means
the inclusion $K\colon \mathbf{A} \hookrightarrow \mathbf{B}$ is full (not required in CWM), and has a left adjoint $L\colon \mathbf{B} \to \mathbf{A}$, called the **localization** or **reflector**.

Example: $K\colon \mathbf{Ab} \hookrightarrow \mathbf{Grp}$
Then $L\colon G \mapsto G/[G,G]$, the largest abelian quotient of G.

Reflective adjunction: $\mathbf{A}(LB, A) \cong \mathbf{B}(B, A)$

Unit: $\eta_B\colon B \to LB$; **counit:** $\varepsilon_A\colon LA \to A$ is an isomorphism.

So, when are counits of adjunctions isomorphisms? Need lemmata:

Lemma 1: $\tau\colon S \to T$ is $\left\{\begin{matrix}\text{epi}\\ \text{mono}\end{matrix}\right\}$ in $\mathbf{Set}^{\mathbf{A}}$ iff $\tau_{A''}$'s $\left\{\begin{matrix}\text{epi}\\ \text{mono}\end{matrix}\right\}$ in $\mathbf{Set}$.

Proof.
$$\begin{array}{ccc} S & \xrightarrow{\tau} & T \\ {\scriptstyle\tau}\downarrow & \text{p-o} & \downarrow{\scriptstyle 1_T} \\ T & \xrightarrow[1_T]{} & T \end{array} \quad \Leftrightarrow \quad \forall\ A'' \in \mathbf{A}_0, \quad \begin{array}{ccc} SA'' & \xrightarrow{\tau_{A''}} & TA'' \\ {\scriptstyle\tau_{A''}}\downarrow & \text{p-o} & \| \\ TA'' & = & TA'' \end{array} \qquad \Box$$

Lemma 2: For $f\colon A' \to A$, natural transformation $R_\circ(f)$ or
$\mathbf{A}(A, f)\colon \mathbf{A}(A, \_) \to \mathbf{A}(A', \_)$ is $\left\{\begin{matrix}\text{mono}\\ \text{epi}\end{matrix}\right\}$ iff f is $\left\{\begin{matrix}\text{epi}\\ \text{split mono}\end{matrix}\right\}$.

[Note $R_\circ(f) \mapsto R_\circ(f)(1_A) = f$ under the Yoneda Lemma.]

Proof. $\mathbf{A}(A, A') \xrightarrow{R_\circ(f)} \mathbf{A}(A', A')$ epi $\Rightarrow \exists\ r \in \mathbf{A}(A, A')\,.\ r \circ f = 1_{A'}$.

Conv., $f^{\mathsf{op}} r^{\mathsf{op}} = 1_{A'} \Rightarrow \forall\ A''\,,\ \exists A''(f^{\mathsf{op}}) \circ \exists A''(r^{\mathsf{op}}) = \exists A''(1_{A'})$
$\Rightarrow R_\circ(f)_{A''} \circ R_\circ(r)_{A''} = 1_{\mathbf{A}(A',A'')} \Rightarrow R_\circ(f)_{A''}$ surj., epi; so $R_\circ(f)$ epi.

$\forall\ A''\,,\ \mathbf{A}(A, A'') \xrightarrow{R_\circ(f)} \mathbf{A}(A', A'')$ mono $\Leftrightarrow h_1 \circ f = h_2 \circ f \Rightarrow h_1 = h_2$ $\Box$.

Theorem: In $\mathbf{A} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{X}$ ($F \dashv U$),

U is …	iff ε_A …
full	has retract
faithful	is epi
full, faithful	is iso

Proof. Natural transformation $\alpha\colon \mathbf{A}(A, \_) \to \mathbf{A}(FUA, \_)$ with component $\alpha_{A'}\colon \mathbf{A}(A, A') \xrightarrow{U_{A,A'}} \mathbf{X}(UA, UA') \xrightarrow{\varphi^{-1}_{UA,A'}} \mathbf{A}(FUA, A')$.
Under Yoneda Lemma, $\alpha \mapsto \alpha_A(1_A) = \varepsilon_A$. Then by Lemma 2:
ε_A split mono $\Leftrightarrow$ α epi $\Leftrightarrow$ $\forall A'\,,\ \alpha_{A'}$ surj. $\Leftrightarrow$ $\forall A'\,,\ U_{A,A'}$ surj;
ε_A epi $\Leftrightarrow$ α mono $\Leftrightarrow$ $\forall A'\,,\ \alpha_{A'}$ mono $\Leftrightarrow$ $\forall A'\,,\ U_{A,A'}$ inj;. $\Box$