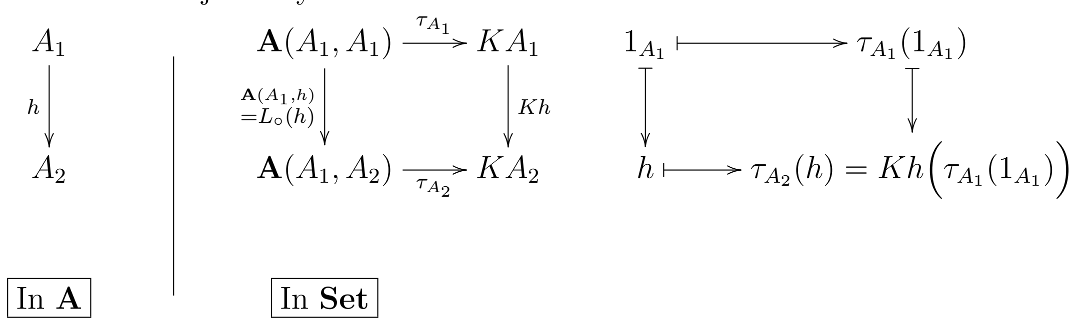
18. The Yoneda Lemma

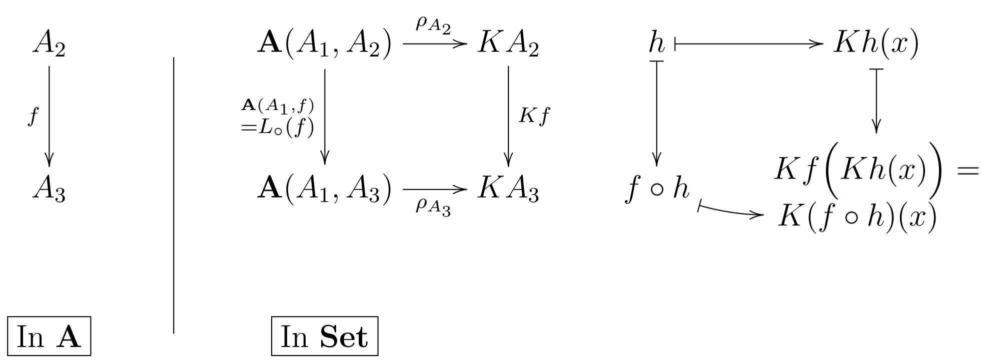
Yoneda Lemma: Let $\mathbf{A}$ be locally small.
For object A_1 of $\mathbf{A}$, and $f\colon A_2 \to A_3$ in $\mathbf{A}_1$, remember
$\mathbf{A}(A_1, f)\colon \mathbf{A}(A_1, A_2) \to \mathbf{A}(A_1, A_3); h \mapsto f \circ h$ post-composes with f.

Then for $K\colon \mathbf{A} \to \mathbf{Set}$, have $\boxed{\mathbf{Set}^{\mathbf{A}}\big(\mathbf{A}(A_1, \_\,), K\big) \cong KA_1; \tau \mapsto \tau_{A_1}(1_{A_1})}$

Proof. • Injectivity:



• Surjectivity, $\rho\colon \mathbf{A}(A_1, A_2) \to K$, $\rho_{A_2}\colon h \mapsto Kh(x)$ for $x \in KA_1$ nat:



□

Corollary: Full, faithful (covariant) **Yoneda embedding**

$$\boxed{\text{ヨ}\colon D \hookrightarrow \widehat{D} = \mathbf{Set}^{D^{\mathsf{op}}}; [f\colon x \to y] \mapsto \big[D(\_\,, f)\colon D(\_\,, x) \to D(\_\,, y)\big]}$$

Category $\widehat{D}$ of (set-valued) **pre-sheaves** over D.

Note: "ヨ" is Katakana for "Yo".

Example: Poset category $(P, \leq)$.
For element x, slice category $D(\_\,, x)$ is (ess.) the down-set $\downarrow x$ of x.
Then for $f\colon x \leq y$,
natural transformation $D(\_\,, f)$ is the inclusion $\downarrow x \hookrightarrow \downarrow y$.