16. Poset adjunctions and Galois correspondences

Poset categories $(\mathbf{A}, \leq)$, $(\mathbf{B}, \leq)$, functors $R\colon \mathbf{A} \to \mathbf{B}$, $S\colon \mathbf{B} \to \mathbf{A}$.

Galois connection: adjunction $\mathbf{A}(Sb, a) \cong \mathbf{B}(b, Ra)$,

$$\text{so } Sb \leq a \;\Leftrightarrow\; b \leq Ra.$$

Unit: $\forall\, b \in B\,,\; b \leq RSb$. **Counit:** $\forall\, a \in A\,,\; SRa \leq a$.

Thus $\forall\, a \in \mathbf{A}\,,\; Ra \leq RSRa$ and $\forall\, b \in \mathbf{B}\,,\; SRSb \leq Sb$ (plug in).

Also $\forall\, b \in \mathbf{B}\,,\; Sb \leq SRSb$ and $\forall\, a \in \mathbf{A}\,,\; RSRa \leq Ra$ (use S, R).

Closed elements: In $S(\mathbf{B}) \subseteq \mathbf{A}$ or $R(\mathbf{A}) \subseteq \mathbf{B}$.

Closure of $a \in \mathbf{A}$ is $SRa = \operatorname{dom} \varepsilon_a$, and of $b \in \mathbf{B}$ is $RSb = \operatorname{cod} \eta_b$.

Galois correspondence: Mut. inverse $\big(S(\mathbf{B}), \leq\big) \underset{S}{\overset{R}{\rightleftarrows}} \big(R(\mathbf{A}), \leq\big)$.

Polarity is a relation $\alpha \subseteq I \times J$. Gives Galois connection

$$S\colon (2^I, \subseteq) \to (2^J, \supseteq); X \mapsto \{y \in J \mid \forall x \in X\,,\; x\,\alpha\,y\}$$

$$R\colon (2^J, \supseteq) \to (2^I, \subseteq); Y \mapsto \{x \in I \mid \forall y \in Y\,,\; x\,\alpha\,y\}$$

Note $\forall\, X \subseteq I\,,\; \forall\, Y \subseteq J\,,$

$$SX \supseteq Y \quad\Leftrightarrow\quad \forall\, x \in X\,,\; \forall\, y \in Y\,,\; x\,\alpha\,y \quad\Leftrightarrow\quad X \subseteq RY.$$

Galois theory: Group permutation representation or G-set (X, G).
Fixed point relation $\{(x, g) \in X \times G \mid gx = x\}$.
Right adjoint $R\colon 2^G \to 2^X$ is the **fixed point functor**.
Left adjoint $S\colon 2^X \to 2^G$ is the (pointwise) **stabilizer functor**.

Polar geometry: Vector space V with quadratic form $\langle \mathbf{u}, \mathbf{v} \rangle$.
Polarity $\{(\mathbf{u}, \mathbf{v}) \mid \langle \mathbf{u}, \mathbf{v} \rangle = 0\} \subseteq V \times V$.
Closure of a subset is its **orthogonal complement**.

Alg. geometry: On $\mathbf{C}^n \times \mathbf{C}[X_1, \ldots, X_n]$, polarity $\{(\mathbf{x}, f) \mid f(\mathbf{x}) = 0\}$.
Closed subsets of $\mathbf{C}^n$ are **algebraic sets** or **varieties**.
Closed subsets of $\mathbf{C}[X_1, \ldots, X_n]$ are **radical ideals**.
Hilbert's Nullstellensatz: The closure of an ideal $\mathfrak{I} \lhd \mathbf{C}[X_1, \ldots, X_n]$ is its **radical** $\sqrt{\mathfrak{I}} = \{f \mid \exists\, 0 < n \in \mathbb{N}\,.\; f^n \in \mathfrak{I}\}$.

Example: Radical of $\langle X_1^2 \rangle$ in $\mathbf{C}[X_1, \ldots, X_n]$ is $\langle X_1 \rangle$.