15. Three adjunctions with monoids

- **Free module functor** $F\colon \mathbf{Set} \to \mathbf{Mon}$ is left adjoint to the forgetful functor $U\colon \mathbf{Mon} \to \mathbf{Set}$.

"Tensor" notation $x_1 \otimes \ldots \otimes x_n$ for n-tuple $(x_1, \ldots, x_n)$.

For set or **alphabet** X, coproduct $FX := \sum\limits_{n\in\mathbb{N}} X^n$, with $X^0 = \{1\}$ and **word concatenation** associative product

$$(x_1\otimes\ldots\otimes x_m, y_1\otimes\ldots\otimes y_n) \mapsto x_1\otimes\ldots\otimes x_m\otimes y_1\ldots\otimes y_n.$$

Note $\lambda_X = 1_X$ gives $1 \otimes x = x$ and similarly $x \otimes 1 = x$ by $\rho_X = 1_X$.

Unit $\eta_X\colon x \mapsto x$ ("alphabet letter makes a one-letter word") and

counit $\varepsilon_M\colon FUM \to M; m_1 \otimes m_2 \mapsto m_1 \cdot m_2$ ("multiplication table").

- **Group of Units functor** $U\colon \mathbf{Mon} \to \mathbf{Grp}; (M, \cdot, 1) \mapsto (M^*, \cdot, ^{-1}, 1)$ is right adjoint to Forgetful $F\colon \mathbf{Grp} \to \mathbf{Mon}; (G, \cdot, ^{-1}, 1) \mapsto (G, \cdot, 1)$.

Natural isomorphism $\varphi_{G,M}\colon \mathbf{Mon}(FG, M) \cong \mathbf{Grp}(G, UM); \theta \mapsto \theta$,

since $g \cdot g^{-1} = 1 \;\Rightarrow \theta(g) \cdot \theta(g)^{-1} = 1$ and dually, so $\theta(g) \in M^*$.

Unit $\eta_G\colon G \to G^*; g \mapsto g$ (note $G^* = G$) and

counit $\varepsilon_M\colon M^* \hookrightarrow M; u \mapsto u$ (embedding group of units into monoid).

- M**-sets for a monoid** M — **categorification** of a monoid M — e.g., permutation representations for M a group.

Functor $L\colon M \to \mathbf{Set}; * \mapsto X,\; m \mapsto [L_m\colon X \to X; x \mapsto mx]$,
can also be written as (X, M), a set X with "scalars" from M,
or as the monoid homomorphism $L\colon M \to \mathbf{Set}(X, X); m \mapsto L_m$.

Category $\mathbf{Set}^M$ of M-**sets**.

Forgetful functor $U\colon \mathbf{Set}^M \to \mathbf{Set}; L \mapsto L(*)$ or $(X, M) \mapsto X$.

Free M**-set functor** $F\colon \mathbf{Set} \to \mathbf{Set}^M; X \mapsto (M \times X, M)$
with $m(n, x) = (mn, x)$.

<u>Free algebra functor is left adjoint to the underlying set functor:</u>

Unit $\eta_X\colon X \to M \times X; x \mapsto (1, x)$
(embedding generators into the free algebra) and

counit $\varepsilon_{(X,M)}\colon (M \times X, M) \to (X, M); (m, x) \mapsto mx$
(action in the M-set).