14. Spaces, bases, adjunctions

Category $\mathcal{L}$ of linear transformations of vector spaces over a field K.

Forgetful or underlying set functor $U \colon \mathcal{L} \to \mathbf{Set}$.

For set X, v. sp. with basis X is $FX = \left\{ \sum_{i=1}^{r} \lambda_i \cdot x_i \mid \lambda_i \in K\,,\ x_i \in X \right\}$,

the space of formal linear combinations $\sum_{i=1}^{r} \lambda_i \cdot x_i$ of elements of X.

At X, have **unit** $\eta_X \colon X \to UFX; x \mapsto 1 \cdot x$ which **inserts** the basis.

At V, have **counit** $\varepsilon_V \colon FUV \to V; \sum_{i=1}^{r} \lambda_i \cdot v_i \mapsto v$, where
$v = \lambda_1 v_1 + \ldots + \lambda_r v_r$, the formal combination **worked out** in V.

For $f \colon X \to Y$, lin. transf. $Ff \colon FX \to FY; \sum_{i=1}^{r} \lambda_i \cdot x_i \mapsto \sum_{i=1}^{r} \lambda_i \cdot f(x_i)$.

So have functors $F \colon \mathbf{Set} \to \mathcal{L}$ and $U \colon \mathcal{L} \to \mathbf{Set}$.

Nat. isom. with components $\varphi_{X,V} \colon \boxed{\mathcal{L}(FX, V) \cong \mathbf{Set}(X, UV)}$ (*)

Mutually inverse $\varphi_{X,V} \colon [FX \xrightarrow{\theta} V] \mapsto [X \xrightarrow{\eta_X} UFX \xrightarrow{U\theta} UV]$
(informally, restricting θ to X); note unit $\eta_X = \varphi_{X,FX}(1_{FX})$;

and dually, $\varphi^{-1}_{X,V} \colon [X \xrightarrow{f} UV] \mapsto [FX \xrightarrow{Ff} FUV \xrightarrow{\varepsilon_V} V]$
(informally, extending f to FX); note counit $\varepsilon_V = \varphi^{-1}_{UV,V}(1_{UV})$.

Adjunction $\boxed{(F, U, \eta, \varepsilon)}$ with **left adjoint** F and **right adjoint** U.

Thus $\varphi_{X,V} \colon \theta \mapsto U\theta \circ \eta_X$ and $\varphi^{-1}_{X,V} \colon f \mapsto \varepsilon_V \circ Ff$.

Triangular identities: $\forall\, X \in \mathbf{Set}_0\,,\ 1_{FX} = \varepsilon_{FX} \circ F\eta_X\ [= \varphi^{-1}_{X,FX}(\eta_X)]$
and $\forall\, V \in \mathcal{L}_0\,,\ 1_{UV} = U\varepsilon_V \circ \eta_{UV}\ [= \varphi_{UV,V}(\varepsilon_V)]$.

The triangular identities are necessary and sufficient for an adjumction.

Other notations: $F \dashv U$ or
$$\begin{tikzcd} \mathcal{L} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathbf{Set} \arrow[l, bend right, "F"'] \end{tikzcd}$$

Mnemonic: In the box (*), put the functors at the extreme edges.
The left adjoint (F) is on the left; the right adjoint (U) is on the right.