12. Cartesian monoidal categories

Cartesian monoidal category: category C with all finite products.

Idea: Think of $(C, \times, \top)$ as like a monoid, say $(\mathbb{N}, +, 0)$ or $(\mathbb{R}, \cdot, 1)$.

Problem of **non-associativity:** e.g. in **Set**, $(x,(y,z)) \neq ((x,y),z)$.

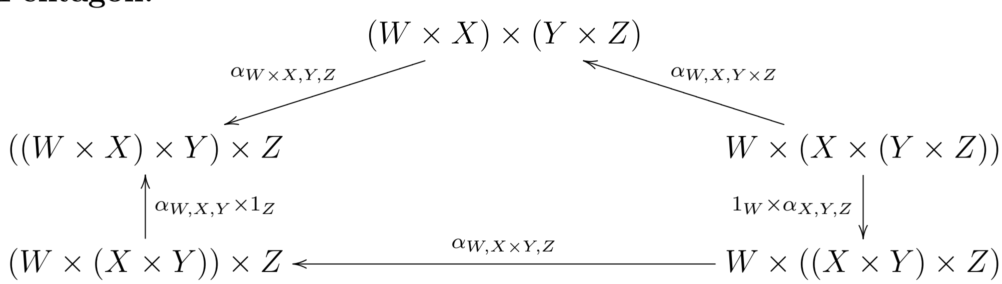
Fix: Bifunctor $C \times C \to C; (X,Y) \mapsto X \times Y$,
trifunctors $C \times C \times C \to C; (X,Y,Z) \mapsto X \times (Y \times Z)$ or $(X \times Y) \times Z$,
nat. isom. α with components $\boxed{\alpha_{X,Y,Z} \colon X \times (Y \times Z) \to (X \times Y) \times Z}$
which commute with projections; both sides give a product of X, Y, Z.
[Typical two-stage projection $\pi_Y \colon X \times (Y \times Z) \xrightarrow{\pi_{Y \times Z}} Y \times Z \xrightarrow{\pi_Y} Y$.]

Problem of **non-unitality:** e.g. in **Set** with $\top = \{*\}$, have $(*, x) \neq x$.

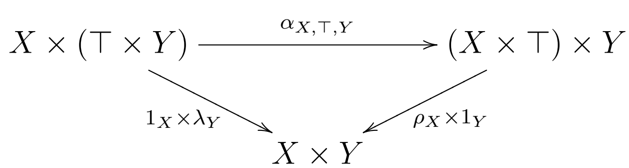
Fix: Functors $C \to C; X \mapsto \top \times X$ or $X \times \top$ nat. isom. to identity,
so components $\boxed{\lambda_X \colon \top \times X \to X}$ and $\boxed{\rho_X \colon X \times \top \to X}$.

Potentially large "monoid" $(C, \times, \top)$ "up to natural isomorphisms".

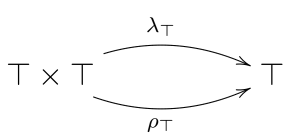
Pentagon:



Triangle:



Digon:



Coherence [CWM §VII.2]: If these three diagrams commute, then w.l.o.g. have a **strict** monoidal category: the nat. isoms. are identities.

- Coherence holds for Cartesian monoidal categories.