

11. PRODUCT CATEGORIES AND BIFUNCTORS

Product $B \times C$ of quivers B, C has $(B \times C)_0 = B_0 \times C_0$,
 $(B \times C)_1 = B_1 \times C_1$, pointwise $\partial_i(f, g) = (\partial_i f, \partial_i g)$ for $i = 0, 1$.

Product $B \times C$ of categories B, C : pointwise identities, composition:
 $(B \times C)((x, x'), (y, y')) \times (B \times C)((y, y'), (z, z'))$
 $\rightarrow (B \times C)((x, x'), (z, z')): ((f, f'), (g, g')) \mapsto (f' \circ f, g' \circ g)$.

Universality: $B \xleftarrow{\pi_B} B \times C \xrightarrow{\pi_C} C$ — graph maps or functors.

$$\begin{array}{ccc} B & \xleftarrow{\pi_B} & B \times C & \xrightarrow{\pi_C} & C \\ & \searrow F & \uparrow F \cap G & & \nearrow G \\ & & D & & \end{array}$$

Example: $B' \xleftarrow{\pi'_B} B' \times C' \xrightarrow{\pi'_C} C'$

$$\begin{array}{ccccc} B' & \xleftarrow{\pi'_B} & B' \times C' & \xrightarrow{\pi'_C} & C' \\ \uparrow F & & \uparrow F \times G & & \uparrow G \\ B & \xleftarrow{\pi_B} & B \times C & \xrightarrow{\pi_C} & C \end{array}$$

Bifunctor S to D on B and C is a functor $S: B \times C \rightarrow D$

— **graph, diagram or quiver bimap** if B, C are just quivers.

Proposition: Given bifunctor $S: B \times C \rightarrow D$: For $(b, c) \in (B \times C)_0$,
define $R_b := S(b, _): C \rightarrow D$ and $L_c := S(_, c): B \rightarrow D$.

Then $\forall f: b \rightarrow b', g: c \rightarrow c'$:

$$\begin{array}{ccc} S(b, c) & \xrightarrow{R_b(g)=S(b,g)} & S(b, c') \\ L_c(f)=S(f,c) \downarrow & \searrow S(f,g) & \downarrow L_{c'}(f)=S(f,c') \\ S(b', c) & \xrightarrow{R_{b'}(g)=S(b',g)} & S(b', c') \end{array}$$

Conversely, given $R_b: C \rightarrow D$ and $L_c: B \rightarrow D$

with $\forall b \in B_0, c \in C_0, L_c(b) = R_b(c)$

and commuting solid square,

the diagonal defines a bifunctor $S: B \times C \rightarrow D$. \square

Example: Locally small $C, B = C^{\text{op}}, R_b: C \rightarrow \mathbf{Set}; b \mapsto C(b, c)$,

$L_c: C^{\text{op}} \rightarrow \mathbf{Set}; b \mapsto C(b, c)$ (like dualizing), $R_b(c) = C(b, c) = L_c(b)$.

For $h \in C(b', c)$, so $b \xrightarrow{f} b' \xrightarrow{h} c \xrightarrow{g} c'$, have

$$\begin{array}{ccc} h \circ f \xrightarrow{R_b(g)} g \circ h \circ f & & C(b, c) \xrightarrow{R_b(g)} C(b, c') \\ \uparrow L_c(f) & & \uparrow L_{c'}(f) \\ h \xrightarrow{R_{b'}(g)} g \circ h & & C(b', c) \xrightarrow{R_{b'}(g)} C(b', c') \end{array}$$