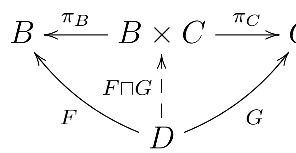
11. Product categories and bifunctors

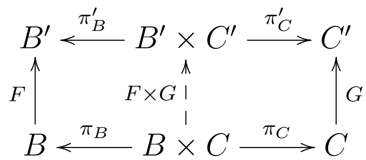
Product $B \times C$ of quivers B, C has $(B \times C)_0 = B_0 \times C_0$, $(B \times C)_1 = B_1 \times C_1$, pointwise $\partial_i(f, g) = (\partial_i f, \partial_i g)$ for $i = 0, 1$.

Product $B \times C$ of categories B, C : pointwise identities, composition:

$$(B \times C)\big((x, x'), (y, y')\big) \times (B \times C)\big((y, y'), (z, z')\big)$$
$$\to (B \times C)\big((x, x'), (z, z')\big) \colon \big((f, f'), (g, g')\big) \mapsto (f' \circ f, g' \circ g).$$

Universality: 

— graph maps or functors.

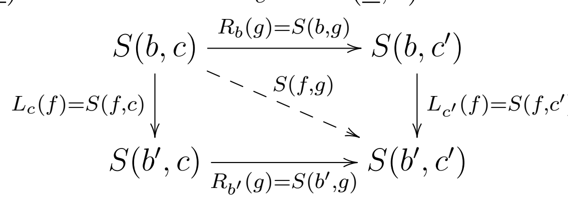
Example: 

Bifunctor S to D on B and C is a functor $S \colon B \times C \to D$

— **graph, diagram** or **quiver bimap** if B, C are just quivers.

Proposition: Given bifunctor $S \colon B \times C \to D$: For $(b, c) \in (B \times C)_0$, define $R_b := S(b, \_) \colon C \to D$ and $L_c := S(\_, c) \colon B \to D$.

Then $\forall\, f \colon b \to b',\ g \colon c \to c'$:



Conversely, given $R_b \colon C \to D$ and $L_c \colon B \to D$
with $\forall b \in B_0, c \in C_0$, $L_c(b) = R_b(c)$
and commuting solid square,
the diagonal defines a bifunctor $S \colon B \times C \to D$. $\square$

Example: Locally small C, $B = C^{\mathsf{op}}$, $R_b \colon C \to \mathbf{Set}; b \mapsto C(b, c)$, $L_c \colon C^{\mathsf{op}} \to \mathbf{Set}; b \mapsto C(b, c)$ (like dualizing), $R_b(c) = C(b, c) = L_c(b)$.

For $h \in C(b', c)$, so $b \xrightarrow{f} b' \xrightarrow{h} c \xrightarrow{g} c'$, have