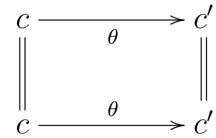
## 10. General limits and colimits

Diagram $D$, category $C$, **constant** or **diagonal** for $\theta \in C(c, c')$ is nat. tr. $\Delta\theta\colon \Delta c \to \Delta c'$ with $\Delta\colon D \to C; [f\colon x \to y] \mapsto$

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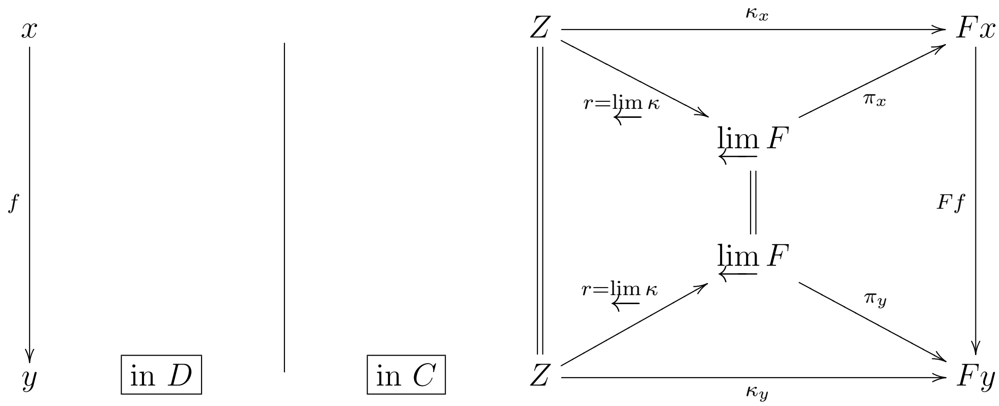
**Limit** of graph map $F\colon D \to C$ is **projection** $\pi\colon \Delta \varprojlim F \to F$ such that $\forall\ \kappa\colon \Delta Z \to F\ ,\ \exists_!\ r = \varprojlim \kappa \in C(Z, \varprojlim F)\ .\ \pi \circ \Delta \varprojlim \kappa = \kappa$.




A.k.a "projective limit" or "inverse limit", written as lim.

**Colimit** of graph map $F\colon D \to C$ is **insertion** $\iota\colon F \to \Delta \varinjlim F$ such that $\forall\ \kappa\colon F \to \Delta Z\ ,\ \exists_!\ r = \varinjlim \kappa \in C(\varinjlim F, Z)\ .\ \Delta \varinjlim \kappa \circ \iota = \kappa$.

A.k.a "inductive limit" or "direct limit", written as colim.

**Example:** Functor (order-preserving) between poset categories $x\colon (\mathbb{N}, \leq) \to (\mathbb{R} \cup \{\infty\}, \leq)\colon n \mapsto x_n$. Then $\varinjlim x = \lim_{n\to\infty} x_n$.

**Example:** $F\colon \mathbb{N} \smallsetminus \{0,1\} \to \mathbf{Ring}; n \mapsto \mathbb{Z}/n\mathbb{Z}$. Then $r = \Delta \varprojlim \kappa\colon \mathbb{Z} \hookrightarrow \varprojlim F = \prod_{n=2}^{\infty} \mathbb{Z}/n\mathbb{Z}$ for $\kappa_n\colon \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}; x \mapsto x + n\mathbb{Z}$.

**Directed diagram** $D$: $\forall\ x, y \in D_0\ ,\ \exists\ z \in D_0\ .\ x \to z \leftarrow y$.

Then have **directed limits** and **directed colimits**.

**Example:** (Real) vector space $V$,
directed poset $\big(\mathcal{P}_{\mathsf{fin}}(V), \subseteq\big)$ of finite subsets.
Functor $F\colon \mathcal{P}_{\mathsf{fin}}(V) \to \mathcal{L}; X \mapsto \mathrm{Span}(X)$. Then $\varinjlim F = V$.

**Theorem:** Each algebra is the (directed) colimit of its finitely generated subalgebras.