9. More limits and colimits

Pullback:
$$
\begin{tikzcd}
Z \arrow[r, dashed, "r"'] \arrow[rr, bend left, "k"] \arrow[dr, bend right, "h"'] & X \times_B Y \arrow[r, "\pi_Y"'] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & B
\end{tikzcd}
$$
or with poset diagram

$\boxed{\bullet \to \bullet \leftarrow \bullet}$ category picture:
$$
\begin{tikzcd}
Z \arrow[rr, "h"] \arrow[dr, dashed, "r"'] \arrow[dd, equal] & & X \arrow[dd, "f"] \\
& X \times_B Y \arrow[ur, "\pi_X"'] \arrow[d, equal] & \\
Z \arrow[r, dashed, "r"] \arrow[rr, bend right, "f\circ h=g\circ k"'] \arrow[dd, equal] & X \times_B Y \arrow[r, "f\circ\pi_X=g\circ\pi_Y"] \arrow[d, equal] & B \\
& X \times_B Y \arrow[dr, "\pi_Y"] & \\
Z \arrow[ur, dashed, "r"] \arrow[rr, "k"'] & & Y \arrow[uu, "g"']
\end{tikzcd}
$$

Ex: Domain of category composition is pullback of $C_1 \xrightarrow{\partial_0} C_0 \xleftarrow{\partial_1} C_1$.
Pushout is the dual of a pullback.

Equalizer:
$$
\begin{tikzcd}
Z \arrow[r, dashed, "r"'] \arrow[rr, bend left, "k"] & E \arrow[r, "e"'] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd}
$$
so $f\circ k = g\circ k \Rightarrow \exists_!\, r\,.\, e\circ r = k$

In $\mathcal{L}$, $E = \mathrm{Ker}(f-g) \overset{e}{\hookrightarrow} X$. In **Set**, $E = \{x \in X \mid fx = gx\} \overset{e}{\hookrightarrow} X$.

Coequalizer:
$$
\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "u"'] \arrow[rr, bend left, "k"] & C \arrow[r, dashed, "r"'] & Z
\end{tikzcd}
$$
; $k\circ f = k\circ g \Rightarrow \exists_!\, r.\, r\circ u = k$

In **Set**, C is quotient of Y by equiv. rel'n. gen. by $\{(fx, gx) \mid x \in X\}$.

In $\mathcal{L}$, u projects from Y to $C = \mathrm{Coker}(f-g) := Y/\mathrm{Im}(f-g)$.

Extended First Isomorphism Theorem in $\mathcal{L}$ is the exact sequence

$$0 \longrightarrow \mathrm{Ker} f \longrightarrow X \xrightarrow{f} Y \longrightarrow \mathrm{Coker} f \longrightarrow 0$$

where **exact** means $\mathrm{Im}\, g_1 = \mathrm{Ker}\, g_2$ for each $\xrightarrow{g_1} \bullet \xrightarrow{g_2}$.

Similar in **Ab**, ${}_R\mathbf{Mod}$, $\mathbf{Mod}_R$, $\mathbf{Mod}_K$ (commutative unital ring K), or any **abelian category** A where each $A(X, Y)$ is an abelian group.