

8. PRODUCTS AND COPRODUCTS

Product $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$ of sets X, Y :

$$\begin{array}{ccccc} X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y \\ & \searrow f & \uparrow f \sqcap g & \swarrow g & \\ & & Z & & \end{array}$$

Universality property: $\forall Z \in \mathbf{Set}_0$, “solid” \downarrow implies \downarrow “dashed”
 bijection $\mathbf{Set}(Z, X) \times \mathbf{Set}(Z, Y) \rightarrow \mathbf{Set}(Z, X \times Y)$; $(f, g) \mapsto f \sqcap g$
 with $f = \pi_X \circ (f \sqcap g)$ and $g = \pi_Y \circ (f \sqcap g)$. Thus $f \sqcap g: z \mapsto (fz, gz)$.

Picture in \mathbf{Set}^2 for discrete “two spot” diagram $2 = \boxed{\bullet \bullet}$:

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ & \searrow f \sqcap g & \nearrow \pi_X \\ & & X \times Y \end{array}$$

$$\begin{array}{ccc} & X \times Y & \\ f \sqcap g \nearrow & & \searrow \pi_Y \\ Z & \xrightarrow{g} & Y \end{array}$$

Examples: Product in \mathbf{Set} carries products in \mathbf{Grp} , \mathbf{Ring} , \mathbf{Mon} , etc.

Example: Product in a poset category is a **greatest lower bound**.

$$\begin{array}{ccc} a & & b \\ \uparrow & \nearrow & \uparrow \\ c & & d \end{array} \quad a \times b = c \text{ exists,}$$

but $c \times d$ does not.

Coproduct in C is the product in C^{op} : $X \xrightarrow{\iota_X} X + Y \xleftarrow{\iota_Y} Y$

$$\begin{array}{ccc} X & \xrightarrow{\iota_X} & X + Y & \xleftarrow{\iota_Y} & Y \\ & \searrow f & \downarrow f \sqcup g & \swarrow g & \\ & & Z & & \end{array}$$

Example: Coproduct in \mathbf{Set} is the disjoint union.

Example: Coproduct in a poset category is a **least upper bound**.

Biproduct $U \xrightleftharpoons[\pi_U]{\iota_U} U \oplus V \xrightleftharpoons[\pi_V]{\iota_V} V$ in \mathcal{L} is product and coproduct.