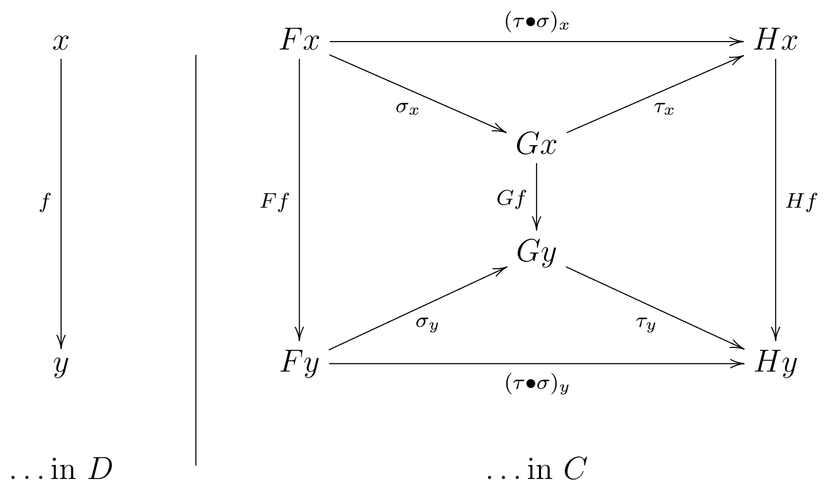
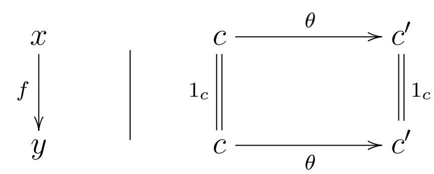
7. Diagram categories and functor categories

Diagram category C^D for diagram D and category C
has graph maps $F, G\colon D \to C$ as objects
and natural transformations $\sigma\colon F \to G$ as morphisms. Composition:



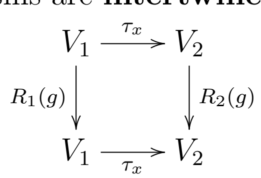
Constant objects and morphisms:



Functor category: Category D, functors $F, G, \ldots$

Example: Linear representations of a group G are objects $R\colon G \to \mathcal{L}$
of the functor category $\mathcal{L}^G$ for the one-object group category G,
so group homomorphisms $R\colon G \to \mathcal{L}(V,V)^* = \operatorname{Aut} V = \mathsf{GL}(V)$.

The morphisms are **intertwiners** or **equivariant maps** $\tau\colon R_1 \to R_2$,
so $\forall\ g \in G$,



E.g., $G = S_3$, $\quad V_1 = \mathbb{R}^3 = \operatorname{Span}\{e_1, e_2, e_3\}$, $\quad R_1(\pi)\colon e_i \mapsto e_{\pi(i)}$.

$V_2 = \mathbb{R}^2 = \operatorname{Span}\{e_2 - e_1, e_3 - e_2\}$, $\quad R_2(\pi)\colon (e_{i+1} - e_i) \mapsto (e_{\pi(i+1)} - e_{\pi(i)})$

$$\overbrace{\frac{1}{3}\begin{bmatrix}-2 & 1 & 1\\ -1 & -1 & 2\end{bmatrix}}^{\tau_x} \cdot \overbrace{\begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}}^{R_1(1\ 2\ 3)} = \overbrace{\begin{bmatrix}0 & -1\\ 1 & -1\end{bmatrix}}^{R_2(1\ 2\ 3)} \cdot \overbrace{\frac{1}{3}\begin{bmatrix}-2 & 1 & 1\\ -1 & -1 & 2\end{bmatrix}}^{\tau_x}$$