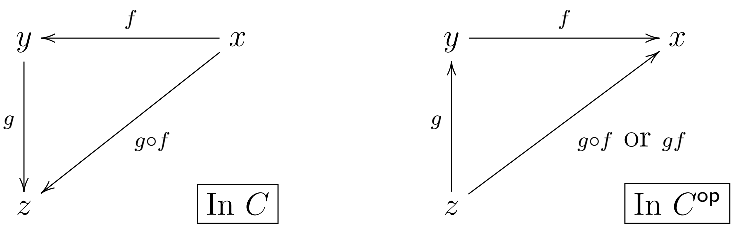
6. Duality and contravariant functors

Dual or **opposite** C^{op} of a category C is built on the dual graph C^{op}: Same identity morphisms, but composition as shown:



For Eulerian notation in C, algebraic notation would be natural in C^{op}.

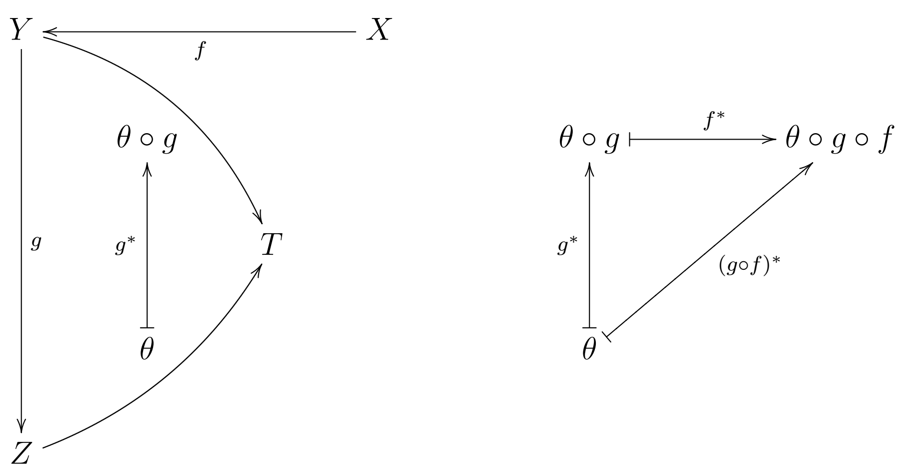
Example: The dual of a one-object monoid category $(M, \cdot, 1_M)$ is the one-object monoid category of the monoid $(M, \circ, 1_M)$ with $x \circ y = y \cdot x$.

Example: For a set X, the dual of the poset category of $\big(\mathcal{P}(X), \subseteq\big)$ is the poset category of $\big(\mathcal{P}(X), \supseteq\big)$.

Contravariant functor $F\colon D \to C$
is a ("covariant" or usual) functor $F\colon D \to C^{\mathsf{op}}$ or $F\colon D^{\mathsf{op}} \to C$.

Thus $F(1_x) = 1_{Fx}$ as usual, but $F(g \circ f) = Ff \circ Fg$.

Generic examples: Locally small C, e.g., **Set** or lin. trans. cat. $\mathcal{L}$. Fix a **dualizing object** $T \in C_0$, e.g., $\mathbf{2} = \{0, 1\} \in \mathbf{Set}_0$ or $\mathbb{R} \in \mathcal{L}_0$. Functor $*\colon C \to \mathbf{Set}^{\mathsf{op}}; Z \mapsto Z^* := C(Z, T)$ with $(g \circ f)^* = f^* \circ g^*$:



From **Set**, set Z^* is the **power set** 2^Z of characteristic functions θ.

From $\mathcal{L}$, vector space Z^* is the **dual space** of linear functionals θ.