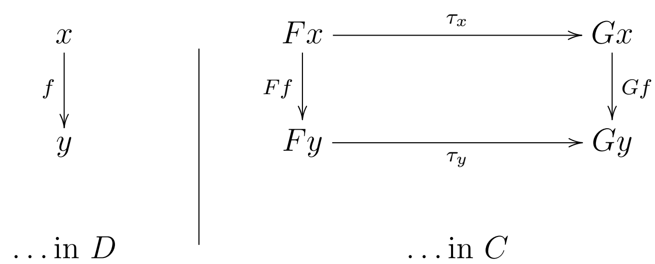
5. Natural transformations

Given graph maps $F, G\colon D \to C$ from a graph D to a category C, a **natural transformation** $\tau\colon F \to G$ is a "vector" $(\tau_x \mid x \in D_0)$ of **components** $\tau_x\colon Fx \to Gx$ in C_1 such that,

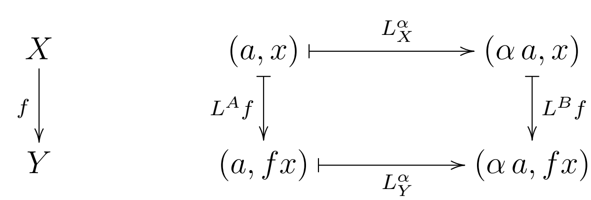
for all $f\colon x \to y$ in D_1, the rectangle of the **naturality diagram**



commutes in the category C.

Natural isomorphism: Each component τ_x is an isomorphism in C.

Example: For a set A, have a functor
$L^A\colon \mathbf{Set} \to \mathbf{Set}; [f\colon X \to Y] \mapsto [A \times X \to A \times Y; (a, x) \mapsto (a, fx)]$.
Then a function $\alpha\colon A \to B$ gives a natural transformation
$L^\alpha\colon L^A \to L^B$ with components $L^\alpha_X\colon A \times X \to B \times X; (a, x) \mapsto (\alpha\, a, x)$
and naturality diagram



Example:
Category $\mathcal{L}$ of (linear transformations between) real vector spaces.
Dual space $V^* = \mathcal{L}(V, \mathbb{R})$ of linear functionals on vector space V.
Double dual $V^{**} = \mathcal{L}(V^*, \mathbb{R}) = \mathcal{L}(\mathcal{L}(V, \mathbb{R}), \mathbb{R})$.
Identity functor $I\colon \mathcal{L} \to \mathcal{L}$.
Double dual functor $DD\colon \mathcal{L} \to \mathcal{L}; V \mapsto V^{**}$.
Natural transformation $\tau\colon I \to DD$ with "evaluation" components

$$\tau_V\colon V \to V^{**}; v \mapsto [\theta \mapsto \theta(v)].$$

Gives a natural isomorphism in finite dimensions.

Contrast: Given basis $\{e_1, \dots e_n\}$ of V, define $\widehat{e}_i\colon V \to \mathbb{R}; e_j \mapsto \delta_{ij}$.

Then $V \to V^*; e_i \mapsto \widehat{e}_i$ does not set up a natural isomorphism.