4. Functors

Functor: A graph homomorphism $F\colon D \to C$, thus with restrictions

$$\forall\ x, y \in D_0\,,\ F_1\colon D(x,y) \to C(F_0x, F_0y)\colon f \mapsto F_1 f\,, \tag{4.1}$$

respecting identities, compositions: $F_1 1_x = 1_{F_0 x}$, $F_1(g \circ f) = F_1 g \circ F_1 f$.

Global conditions: $\begin{cases} \textbf{Isomorphism: } F_0 \text{ and } F_1 \text{ are isomorphisms.} \\ \textbf{Essentially surjective: } \forall\ c \in C_0\,,\ \exists\ d \in D_0\,.\ c \cong F_0 d \end{cases}$

Local conditions: $\begin{cases} \textbf{Full: } \text{Each restriction (4.1) is surjective.} \\ \textbf{Faithful: } \text{Each restriction (4.1) is injective.} \end{cases}$

Example: While the **forgetful** or **underlying set functor**

$$U\colon \mathbf{Grp} \to \mathbf{Set}; [f\colon (G_1, \cdot, {}^{-1}, 1) \to (G_2, \cdot, {}^{-1}, 1)] \mapsto [f\colon G_1 \to G_2]$$

is faithful, $U_0\colon \mathbf{Grp}_0 \to \mathbf{Set}_0$ is not injective (same set, different groups). Also U not full: Some functions between groups are not homomorphic.

Example: For a monoid $(M, \cdot, 1_M)$, write M^* for the group of invertible elements or **units**. Then $\mathbf{Mon} \to \mathbf{Grp}$;

$$[f\colon (M_1, \cdot, 1) \to (M_2, \cdot, 1)] \mapsto [f|_{M_1^*}\colon (M_1^*, \cdot, {}^{-1}, 1) \to (M_2^*, \cdot, {}^{-1}, 1)]$$

is a functor between large categories, the **group of units** functor.

Moral: | Mathematical constructions are functors! |

Example: A monoid homomorphism $f\colon M_1 \to M_2$ yields a functor between the corresponding small one-object categories.

Note $(\mathbb{R}, \cdot, 1) \to ([0, \infty[, \cdot, 1); n \mapsto n^2$ is full, but not faithful.

Example: A functor $F\colon (P_1, \leq) \to (P_2, \leq)$ between poset catgeories corresponds to an **order-preserving** function:

$$x \leq y \ \text{ in } \ P_1 \quad \Rightarrow \quad F_0 x \leq F_0 y \ \text{ in } \ P_2\,.$$

Trivially faithful.

Example: Inclusion of a subcategory always gives a faithful functor.

Full subcategory: The inclusion functor is full.

Example: Category **FinSet** of finite sets is full in **Set**.

Example: Functor

$(\mathbb{N}, \leq) \to \mathbf{FinSet}; [n < n{+}1] \mapsto [\{0, 1, \ldots, n{-}1\} \hookrightarrow \{0, 1, \ldots, n{-}1, n\}]$

is essentially surjective.