3. Special morphisms and objects

Consider morphism $f\colon x \to y$.

- **Isom. or invertible:** $\exists\, f'\colon y \to x\,.\, f \circ f' = 1_y$ and $f' \circ f = 1_x$.
  Ex: Bijective function in **Set**.
- **Monomorphism:** $\forall\, g_i\colon z \to x\,,\ f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$.
  Ex: Injective function in **Set**.
- **Epimorphism:** $\forall\, g_i\colon y \to z\,,\ g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2$.
  Ex: Surjective function in **Set**.
- **Retract** or **split epimorphism:** $r\colon y \to x$ with $r \circ f = 1_x$.
  Ex: $r\colon n \mapsto \max\{0, n-1\}$ retracts successor function on $\mathbb{N}$.
- **Section** or **split monomorphism:** $s\colon y \to x$ with $f \circ s = 1_y$.
  Ex: Successor function on $\mathbb{N}$ is a section of $r\colon \mathbb{N} \to \mathbb{N}$.
- **Idempotent:** $x = \partial_0 f = \partial_1 f$ and $f \circ f = f$.
  Ex: $\mathbb{R}^2 \to \mathbb{R}^2; (x_1, x_2) \mapsto (x_1, 0)$ in **Set**.

Lemma. For $x \underset{s}{\overset{r}{\rightleftarrows}} y$ with $r \circ s = 1_y$:

- r is an epimorphism.
- s is a monomorphism.
- $s \circ r$ is an idempotent (said to **split**).

Isomorphic objects $x \cong y$: Have isomorphism $f\colon x \to y$.

Terminal object $\top$ for C has $\forall\, x \in C_0\,,\ |C(x, \top)| = 1$.

Examples: $\{0\}$ in **Set**, upper bound in a poset, ...

Initial object $\bot$ for C has $\forall\, x \in C_0\,,\ |C(\bot, x)| = 1$.

Examples: $\varnothing$ in **Set**, lower bound in a poset, $\mathbb{Z}$ in **Ring**, ...

Zero object 0 for C is both initial and terminal.

Examples: $\{0\}$ in categories of groups, vector spaces, ...

Groupoid: Category where all morphisms are invertible.

Examples: For a set X:

- **Discrete category** $(X, \{1_x \mid x \in X\}, \partial_0\colon 1_x \mapsto x, \partial_1\colon 1_x \mapsto x)$.
- **Symmetric group** $X!$ of all bijections $X \to X$.
- The collection $\operatorname{Inv} X$ of all bijections between subsets of X.