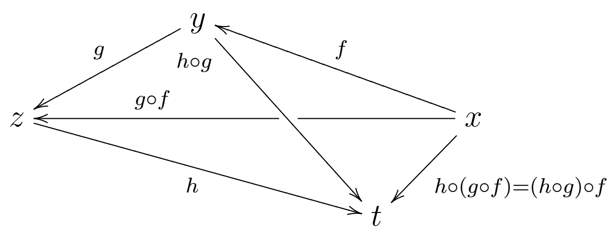
2. Categories

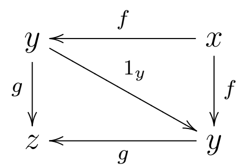
Category: Quiver $C = (C_0, C_1, \partial_0 \colon C_1 \to C_0, \partial_1 \colon C_1 \to C_0)$ with:

- **composition:** $\forall\ x, y, z \in C_0$,
$C(x,y) \times C(y,z) \to C(x,z); (f,g) \mapsto g \circ f$

- satisfying **associativity:** $\forall\ x, y, z, t \in C_0$,
$\forall\ (f,g,h) \in C(x,y) \times C(y,z) \times C(z,t)\,,\ h \circ (g \circ f) = (h \circ g) \circ f$



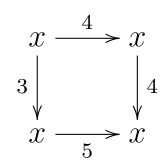
- **identities:** $\forall\ x, y, z \in C_0\,,\ \exists\ 1_y \in C(y,y)\,.$
$\forall\ f \in C(x,y)\,,\ 1_y \circ f = f$ and $\forall\ g \in C(y,z)\,,\ g \circ 1_y = g$



Example: $\mathbb{N}_0 = \{x\}\,,\ \mathbb{N}_1 = \mathbb{N}\,,\ 1_x = 0\,,\ \forall\, m, n \in \mathbb{N}\,,\ n{\circ}m = m{+}n\,;$ —

one object, lots of arrows [**monoid** of natural numbers under addition]

Equation: $3 + 5 = 4 + 4$ **Commuting diagram:**



Example: $\mathbb{N}_1 = \mathbb{N}\,,\ \forall\ m, n \in \mathbb{N}\,,\ |\mathbb{N}(m,n)| = \begin{cases} 1 & \text{if } m \le n; \\ 0 & \text{otherwise} \end{cases}$

— lots of objects, lots of arrows [poset $(\mathbb{N}, \le)$ as a category]

These two examples are **small categories:** have a set of morphisms.

Example: The category **Set** has the class of all sets as its object class, with $\mathbf{Set}(X,Y)$ as the set of all functions from X to Y, composition of functions: $g \circ f(x) = g\big(f(x)\big)$, usual identities $1_X \colon X \to X; x \mapsto x$.

This example is **large** (not small), but **locally small:**
just a set of arrows between each pair of objects.