MATH 584 FALL 2002 TAKE-HOME FINAL

No more than 6 answers will be graded: each worth 7 points, maximum 40.

(1) Let Ob(**Rel**) be the class of sets. For sets A and B, let $\mathbf{Rel}(A, B)$ be the set of all subsets of $A \times B$. For $R \subseteq A \times B$ and $S \subseteq B \times C$, let $S \circ R = \{(a, c) \in A \times C \mid \exists b \in B \;.\; (a, b) \in R \text{ and } (b, c) \in S\}$. Show that **Rel** is a category.

(2) Let **Twoup** be the full subcategory of the category of sets comprising all sets with two or more elements. Show that **Twoup** has no initial object and no terminal object.

(3) Construe a non-trivial group as a category **G** with a single object, and with morphism set equal to the set of elements of the group. Show that the category **G** does not have products.

(4) Show that the poset category of divisors of 12 is isomorphic to a category of sets and functions.

(5) Suppose that a category **C** has a terminal object, and all pullbacks. Show that **C** has all equalizers.

(6) Let **C** be a category in which each morphism is a monomorphism, and for which there are two distinct morphisms having the same domain and the same codomain. Show that there are objects A and B of **C** for which the product $A \times B$ does not exist.

(7) Let **C** be the category of complex vector spaces, and let **R** be the category of real vector spaces. Let $G : \mathbf{C} \to \mathbf{R}$ be the forgetful functor that forgets the non-real scalar multiplications. Show that G has a left adjoint F.

(8) Consider the functor $S : \mathbf{Set} \to \mathbf{Set}$ with $SA = A \times A$ and

$$Sf : A \times A \to B \times B; (a, a') \mapsto (f(a), f(a'))$$

for a function $f : A \to B$. Show that S is naturally isomorphic to the functor $\mathbf{Set}(\mathbf{2}, \_\,)$, where $\mathbf{2} = \{0, 1\}$.

(9) In a category **C**, an epimorphism $u : B \to E$ is *regular* if it is the coequalizer of a pair of arrows $f, g : A \to B$. Show that in the category **Set** of sets, each epimorphism is regular.

(10) Prove that the category of finite-dimensional real vector spaces is equivalent to the category $\mathbf{Matr}_{\mathbb{R}}$ of real matrices.