MATH 510 SPRING 2003 MID-TERM

Hand in 3 answers for grading. Each question is worth $8\frac{1}{3}$ points.

(1) Let A be an invertible $n \times n$ matrix with complex coefficients. Show that there is a complex polynomial $q(t)$, of degree less than n, such that $q(A) = A^{-2}$.

(2) Recall that a matrix A in $\mathrm{M}_n(\mathbb{C})$ is said to be *Hermitian* if $A^* = A$ and *skew-Hermitian* if $A^* = -A$.

  (a) Show that each element A of $\mathrm{M}_n(\mathbb{C})$ may be expressed uniquely as a sum

$$A = A_H + A_K$$

  of an Hermitian matrix A_H and a skew-Hermitian A_K.

  (b) Show that A is normal iff A_H commutes with A_K.

(3) Prove that in $\mathrm{M}_2(\mathbb{R})$, the matrix

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

is not similar to an upper triangular matrix.

(4) Let A be an $n \times n$ matrix with complex coefficients. Show that linear independence of the set $\{A, A^2, \ldots, A^n\}$ implies that A is invertible.

(5) Explain carefully what will go wrong if you try to apply the Gram-Schmidt Orthonormalization Process to a linearly dependent set of vectors in $\mathbb{C}^n$.