# MATH 504 FALL 2003 PRACTICE MID-TERM

*Hand in 3 answers for grading. Each question is worth $8\frac{1}{3}$ points.*

(1) For a set function $f : A \to B$, suppose that there are set functions $g : B \to A$ and $h : B \to A$ such that $fg = 1_A$ and $hf = 1_B$. Prove $g = h$.

(2) For non-empty sets $X, Y$, let $P(x, y)$ be a proposition involving elements $x$ of $X$ and $y$ of $Y$. Define $E(y) = (\exists x \in X.\ P(x, y))$. Prove the following equality between truth values:

$$[E(y)] = \max\{[P(x, y)] \mid x \in X\}$$

(3) Let $A$ be a set. Prove that the direct power $A^n$ is isomorphic to $\underline{\underline{\mathrm{Set}}}(\{1, 2, \ldots, n\}, A)$ for each positive integer $n$.

(4) Let $\alpha$ be a reflexive and transitive relation on a set $A$. Define a relation $\beta$ on $A$ by

$$x\ \beta\ y \qquad \Leftrightarrow \qquad (x\ \alpha\ y \text{ and } y\ \alpha\ x).$$

(a) Prove that $\beta$ is an equivalence relation on $A$.

(b) Prove that

$$x^\beta \leq y^\beta \qquad \Leftrightarrow \qquad x\ \alpha\ y$$

yields a well-defined order relation on the quotient $A^\beta$.

(5) An element $x$ of a monoid $M$ is *invertible* if and only if its image $R_x : M \to M$ under the right regular representation $R : M \to M^M$ is an invertible function.

(a) Show that the set $M^*$ of invertible elements of $M$ forms a submonoid of $M$.

(b) Prove $x \in M^* \Rightarrow \exists y \in M.\ xy = 1_M$.