MATH 302A SPRING 2014 GRADED HOMEWORK #1

Write clearly. Credit is given for the best three answers.

(1) Let R be a commutative, unital ring. Define
$$R[\omega] = \left\{ \begin{bmatrix} x & y \\ -y & x-y \end{bmatrix} \in R^2_2 \;\middle|\; x, y \in R \right\}.$$
Show that $R[\omega]$ forms a commutative, unital ring under the usual matrix addition and multiplication.

(2) If $a\mathbb{Z} \cap b\mathbb{Z} = c\mathbb{Z}$ for positive integers a, b, and c, express the integer c in terms of a and b. Justify your answer.

(3) Show that
$$\sum_{r=0}^{n} 2^r \binom{n}{r} = 3^n$$
for each natural number n.

(4) Let R be a commutative unital subring of a unital ring S.
 (a) Show that the set S^S of functions from S to S forms a unital ring under pointwise addition and multiplication.
 (b) Consider the function $\mathrm{ev}\colon R[X] \to S^S$ that assigns the polynomial function $S \to S; c \mapsto p(c)$ to each polynomial $p(X) \in R[X]$. Show that ev is a unital ring homomorphism.
 (c) Show that two polynomials $p(X)$, $q(X)$ in $R[X]$ yield the same polynomial function $S \to S$ if and only if they lie in the same coset of $\operatorname{Ker}\mathrm{ev}$.

(5) Let R be a nontrivial unital subring of an integral domain S.
 (a) Show that $R \cap S$ is an integral domain.
 (b) Show that $R \times S$ is not an integral domain.