MATH 301B FALL 2009 PRACTICE FINAL

Write clearly, on separate paper. All questions carry equal weight.
You will receive credit for your five best answers.

(1) Let $f : X \to Y$ be a function, with nonempty domain. Show that there is a function $g : Y \to X$ such that $f \circ g \circ f = f$.

(2) Consider the figure

Writing each element as a product of disjoint cycles, determine
(a) the group of symmetries of the figure in 2-space and
(b) the group of symmetries of the figure in 3-space.

(3) Let d be a positive integer. Show that the residue of an integer n modulo d is a unit of the monoid $(\mathbb{Z}/_d, \cdot, 1)$ if and only if n is coprime to d.

(4) Let G be a group, with subgroup H and normal subgroup N.
(a) Show that NH is a subgroup of G.
(b) Show that NH/N is a subgroup of G/N

(5) Define
$$G = \left\{ A \in (\mathbb{Z}/_2)^2_2 \;\middle|\; \det A \neq 0 \right\}$$
and
$$X = \left\{ \mathbf{v} \in (\mathbb{Z}/_2)^1_2 \;\middle|\; \mathbf{v} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}.$$
(a) Show that G forms a group under multiplication. [You may assume that $\det(AB) = \det A \cdot \det B$ for A, B in $(\mathbb{Z}/_2)^2_2$.]
(b) Show that
$$\lambda : G \to X!; A \mapsto (\mathbf{v} \mapsto A\mathbf{v})$$
is a group homomorphism.

(6) Let a and b be positive integers. Show that
$$a\mathbb{Z} + b\mathbb{Z} = \gcd(a, b)\mathbb{Z}\,.$$