MATH 3010-1 SPRING 2025 PRACTICE FINAL

Write clearly, on separate paper. All questions carry equal weight.
You will receive credit for your five best answers.

(1) Prove or disprove the following statement:
   A function $f : X \to Y$ is injective if and only if there is a function $r : Y \to X$ such that $r \circ f = \mathrm{id}_X$.

(2) For a positive integer d, define $\varphi(d) = |(\mathbb{Z}/_d, \cdot, 1)^*|$.
   (a) Show that $\varphi(p) = p - 1$ if p is prime.
   (b) Show that $\varphi(mn) = \varphi(m)\varphi(n)$ if m and n are coprime.

(3) Let e be an element of a group $(G, \cdot)$.
   (a) Show that the set G forms a group under the multiplication

$$*\colon G \times G \to G; (x, y) \mapsto xe^{-1}y.$$

   (b) Show that the group $(G, *)$ with multiplication given in (a) is isomorphic with the original group structure $(G, \cdot)$ on G.

(4) Let X be a subset of a ring R. Let L be the intersection of all the ideals of R that contain X. Show that L is an ideal of R.

(5) Let X be a subset of a ring R. Show that

$$C = \{r \text{ in } R \mid rx = xr \text{ for all } x \text{ in } X\}$$

is a subring of R.

(6) Let x be an element of a unital ring R. If $x^7 = 0$, show that $1 - x$ is an invertible element of $(R, \cdot, 1)$.

(7) Find the minimal polynomial of the kernel ideal of the evaluation homomorphism

$$\mathbb{Z}/_5[X] \to (\mathbb{Z}/_5)^2_2; p(X) \mapsto p\left(\begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}\right).$$