

## COMMENTS ON MATHEMATICS 201

In these comments, (BP) refers to the *Book of Proof*, while (BA) refers to *Basic Analysis*.

**Natural numbers.** Both textbooks adopt a rather old-fashioned approach to the concept of a natural number, identifying it with a positive integer. It is better to follow the modern convention, defining natural numbers as the cardinalities of finite sets. This includes zero as the cardinality of the empty set. Then one may draw analogies between the arithmetic of sets and the arithmetic of natural numbers. Another clear advantage at this level is the ability to take easier base cases in inductions.

**Implications.** The book (BP) uses the symbol  $\Rightarrow$  for implications, both in formal logic and in writing mathematical text. It is better to use the symbol  $\rightarrow$  for formal logic, and reserve  $\Rightarrow$  for use in text, much as  $\wedge$  is used in formal logic, while the word “and” (or occasionally, the symbol  $\&$ ), is used in mathematical text.

**Equivalence relations and modular arithmetic.** The book (BP) uses divisibility statements for integers as a source of test cases for practising proofs. Many of these proofs become trivial when modular arithmetic is used. It should be made clear that modular arithmetic is to be regarded as a higher-level theory that is not available for the purposes of these proofs in Math 201. Modular arithmetic, along with a careful and comprehensive treatment of equivalence relations, including issues of defining operations in terms of equivalence class representatives, is a topic for Math 301.

**Binomial coefficients.** In (BP), Definition 3.2, the binomial coefficient  $\binom{n}{k}$  is defined combinatorially, as the number of  $k$ -element subsets of an  $n$ -element set. For the purposes of Math 201, where there is very little time to cover a broad range of material, it is better to define the binomial coefficients as the coefficients in the Binomial Theorem:

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k,$$

leaving a fuller treatment of the combinatorial definition to Math 304. The combinatorial property may be mentioned as a consequence of the Binomial Theorem, in terms of the positions of the  $n$  factors  $(x + y)$  for which the summand  $y$  is chosen.

**Strong induction.** In §10.1, the book (BP) describes “strong induction” as a proof technique that is separate from induction. This is an unnecessary complication, and should be avoided. What (BP) calls “strong induction” for proof of a statement  $P(n)$  parametrized by a natural number  $n$  is merely a usual inductive proof of the statement  $Q(n) \equiv P(0) \wedge P(1) \wedge \dots \wedge P(n)$ . It is important to learn to be flexible in formulating induction hypotheses.

**Completeness of  $\mathbb{R}$ .** The book (BA) axiomatizes the completeness of  $\mathbb{R}$  by declaring that subsets bounded above have suprema (the so-called *least-upper-bound property*). But it then goes on to use infima of subsets bounded below, without any explicit justification of why they should exist. This is very confusing. There are two short-term ways to fix this problem as it arises in §1.2:

- At least mention, and possibly consider as a rather difficult exercise, the purely order-theoretical proof that the existence of the suprema implies the existence of the infima (the infimum of a set bounded below is the supremum of its set of lower bounds);
- Use the abelian group structure of  $\mathbb{R}$  to show that the existence of the suprema implies the existence of the infima, namely

$$\inf E = -\sup(-E)$$

as in Proposition 1.2.6(vi).

In the longer term, the order-theoretical characterization of completeness is a dead end (for analysis), since it does not lend itself to higher-dimensional spaces. For this reason, it is better to introduce the convergence of Cauchy sequences as a completeness axiom, replacing the order-theoretical axiom. Indeed, this approach enables one to skip §2.3, which cannot be covered properly at this level in the time available. Thus the hard “only if” direction of Theorem 2.4.5 merely becomes the new completeness axiom.