34. The Comparison Test

Proposition. Suppose that $\forall\ h \le n \in \mathbb{N}\,,\ 0 \le x_n \le y_n$.
(a) If $\sum y_n$ converges, then so does $\sum x_n$.
(b) If $\sum x_n$ diverges, then so does $\sum y_n$.

Proof. (a) Suppose $\sum y_n = Y$. Then the partial sums of $\sum x_n$ form a monotonic increasing sequence bounded above by Y, and so converge.

(b) The partial sums of $\sum x_n$ form a monotonic increasing sequence which does not converge. Thus the sequence is unbounded. Since $\forall\ h \le k \in \mathbb{N}$, we have

$$\sum_{n=h}^{k} x_n \le \sum_{n=h}^{k} y_n\,,$$

the sequence of partial sums of $\sum y_n$ is also unbounded, and therefore does not converge. □

The Riemann Zeta function. This is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}\,, \qquad \text{e.g.,} \qquad \zeta(2) = \frac{\pi^2}{6} \simeq 1.645\,.$$

The series converges for $s > 1$:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^s} &= 1 + \left(\frac{1}{2^s} + \frac{1}{3^s}\right) + \left(\frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s}\right) + \dots \\
&\le 1 + \left(\frac{1}{2^s} + \frac{1}{2^s}\right) + \left(\frac{1}{4^s} + \frac{1}{4^s} + \frac{1}{4^s} + \frac{1}{4^s}\right) + \dots \\
&= 1 + \left(\frac{2}{2^s}\right) + \left(\frac{4}{4^s}\right) + \dots \\
&= \frac{1}{1^{s-1}} + \frac{1}{2^{s-1}} + \frac{1}{4^{s-1}} + \dots \\
&= \left(\frac{1}{2^{s-1}}\right)^0 + \left(\frac{1}{2^{s-1}}\right)^1 + \left(\frac{1}{2^{s-1}}\right)^2 + \dots \\
&= \frac{1}{1 - 2^{1-s}} = \frac{2^{s-1}}{2^{s-1} - 1}\,.
\end{aligned}$$