33. Series

Definition. For $h \in \mathbb{N}$, a *series* or *infinite series*

$$\sum_{n=h}^{\infty} x_n \qquad \text{or informally} \qquad x_h + x_{h+1} + x_{h+2} + \dots$$

means the sequence

$$\left\{ \sum_{n=h}^{k} x_n \right\}_{h \le k \in \mathbb{N}}$$

of *partial sums*

$$\sum_{n=h}^{k} x_n$$

of the *summands* x_n. Then write

$$\sum_{n=h}^{\infty} x_n = L$$

if the sequence of partial sums converges to L.

Example: Euler's constant $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.7182818284\dots$.

Most of the sequence terminology carries over, so have "convergent series," "bounded series," "divergent series," "Cauchy series," etc.

Special series. Some series are easy to handle.

Geometric series:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \qquad \text{for "ratio" } r \text{ with } |r| < 1\,.$$

Telescoping series: Given a convergent sequence $\{y_n\}_{h \le n \in \mathbb{N}} \to y$,

$$\sum_{n=h}^{\infty} (y_n - y_{n+1}) = y_h - y$$

since

$$\begin{aligned} \sum_{n=h}^{k} (y_n - y_{n+1}) &= (y_h - y_{h+1}) + (y_{h+1} - y_{h+2}) + \dots + (y_k - y_{k+1}) \\ &= y_h - y_{k+1} \to y_h - y\,. \end{aligned}$$