32. Cauchy sequences

Definition. The sequence $\{x_n\}_{n\in U}$ is *convergent* if

$$\exists\, L \in \mathbb{R}\,.\ \forall\, \varepsilon > 0\,,\ \exists\, M \in \mathbb{N}\,.\ \forall\, M \le n \in U\,,\ |x_n - L| < \varepsilon\,.$$

— 4 quantifiers, compares terms against some limit L.

Definition. The sequence $\{x_n\}_{n\in U}$ is a *Cauchy sequence* if

$$\forall\, \varepsilon > 0\,,\ \exists\, M \in \mathbb{N}\,.\ \forall\, M \le m, n \in U\,,\ |x_m - x_n| < \varepsilon\,.$$

— 3 quantifiers, compares terms against each other.

Proposition. A convergent sequence is a Cauchy sequence.

Proof estimate:

$$\begin{aligned} |x_m - x_n| &= |(x_m - L) + (L - x_n)| \\ &\le |x_m - L| + |L - x_n| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,. \quad \square \end{aligned}$$

Proposition. A Cauchy sequence is bounded.

Proof. For $\{x_n\}_{n\in U}$, choose $M \in U$ so $\forall\, M \le m, n \in U\,,\ |x_m - x_n| < 1$. Then $\forall\, k \in U\,,\ |x_k| \le \max\big\{1 + |x_M|, \max\{|x_l| \mid M > l \in U\}\big\}$. $\square$

Completeness of $\mathbb{R}$. Want to know that Cauchy sequences converge:

Foundational approach: Replace the l.u.b. property for $\mathbb{R}$ by the *Cauchy completeness*: Every Cauchy sequence converges.

Halving approach: Cauchy sequence $\{x_n\}_{n\in U}$ bounded by $B > 0$. Infinitely many terms in left or right half of $[-B, B]$. Choose. Then infinitely many terms in left or right half of previously chosen half. Choose. Iterate …

Textbook approach: Use "lim sup," "lim inf."