31. Continuity of functions

Definition. Consider a function $f\colon D \to \mathbb{R}$.

(a) The function is *continuous at a point* c in D if, whenever $\lim c_n = c$ for a sequence of points c_n in D, then $\lim f(c_n) = f(c)$. Otherwise, it's *discontinuous* at c.

(b) The function is *continuous* if it continuous at each point c in D.

Example. The identity function $\mathrm{id}_{\mathbb{R}}\colon \mathbb{R} \to \mathbb{R}$ is (trivially) continuous.

Example. The reciprocal function $\mathbb{R} \smallsetminus \{0\} \to \mathbb{R}; x \mapsto 1/x$ is continuous, since $\lim c_n = c$ implies $\lim 1/c_n = 1/c$ for $c \neq 0$.

Example. The function

$$\mathbb{R} \to \mathbb{R}; x \mapsto \begin{cases} 1 & \text{for } x > 0; \\ 0 & \text{for } x \leq 0 \end{cases}$$

is discontinuous at 0. Have $1/n \to 0$, but $f(1/n) = 1 \to 1 \neq 0 = f(0)$.

Proposition. A function $f\colon D \to \mathbb{R}$ is continuous at c in D if and only if the condition

$$\forall\, \varepsilon > 0\,,\ \exists\, \delta > 0\,.\ \forall\, x \in D\,,\ |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

holds.

Properties of continuity. We summarize a couple of typical results.

Proposition. If $f\colon C \to D \subseteq \mathbb{R}$ and $g\colon D \to \mathbb{R}$ are continuous, then so is $g \circ f\colon C \to \mathbb{R}$.

Proof. For $c \in C$: $\lim c_n = c \Rightarrow \lim f(c_n) = f(c)$ — (f is continuous), $\lim f(c_n) = f(c) \Rightarrow \lim g\big(f(c_n)\big) = g\big(f(c)\big)$ — (g is continuous). □

Proposition. Suppose $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ are continuous at $c \in D$.
Then so are $f(x) + g(x)$, $f(x)g(x)$, and $f(x)/g(x)$, provided $g(c) \neq 0$ in the latter case.

Corollary. Polynomials are continuous.