30. Inequalities and algebra for limits

Squeezing Lemma. Consider sequences $\{a_n\}_{n\in U}$, $\{x_n\}_{n\in U}$, $\{b_n\}_{n\in U}$. Suppose

$$\forall\, n \in U\,,\ a_n \le x_n \le b_n$$

and $\lim a_n = \lim b_n = L$. Then $\lim x_n = L$.

Proof. Suppose $\forall\, \varepsilon > 0\,,\ \exists\, M_1\,.\ \forall\, M_1 \le n \in U\,,\ L-\varepsilon < a_n < L+\varepsilon$
and $\forall\, \varepsilon > 0\,,\ \exists\, M_2\,.\ \forall\, M_2 \le n \in U\,,\ L-\varepsilon < b_n < L+\varepsilon$.
Take $M = \max\{M_1, M_2\}$. Then given a tolerance $\varepsilon > 0$, have

$$\forall\, M \le n \in U\,,\ L-\varepsilon < a_n \le x_n \le b_n < L+\varepsilon\,,$$

so $\lim x_n = L$. $\square$

Algebra for limits. Consider convergent sequences $\{a_n\}_{n\in U}$ and $\{b_n\}_{n\in U}$.

(1) $\lim(a_n + b_n) = \lim a_n + \lim b_n$
(2) $\lim(a_n b_n) = (\lim a_n)\cdot(\lim b_n)$
(3) $\lim(1/b_n) = 1/(\lim b_n)$ if $\lim b_n \neq 0$ and $\forall\, n \in U\,,\ b_n \neq 0$

Set $\lim a_n = a$ and $\lim b_n = b$.

Estimate for (1):

$$|(a_n + b_n) - (a+b)| = |(a_n - a) + (b_n - b)| \le |a_n - a| + |b_n - b|$$

Estimate for (2):

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &\le |a_n - a|\cdot|b_n| + |a|\cdot|b_n - b| \\ &\le |a_n - a|\cdot B + |a|\cdot|b_n - b| \end{aligned}$$

with $\forall\, n \in U\,,\ |b_n| \le B$.