29. Subsequences

Definition. A *subsequence* $\{x_n\}_{n\in S}$ of a sequence $\{x_n\}_{n\in U}$ is a function

$$S \to \mathbb{R}; n \mapsto x_n$$

whose domain is a countably infinite subset S of U.

Proposition. If $\{x_n\}_{n\in U}$ converges to L,
then each subsequence $\{x_n\}_{n\in S}$ converges to L.

Fact: A divergent sequence may have a convergent subsequence.

Tails. Let $\{x_n\}_{n\in U}$ be a sequence.

Definition. A subsequence $\{x_n\}_{n\in T}$ of a sequence $\{x_n\}_{n\in U}$ is a *tail* if $U \smallsetminus T$ is finite.

Proposition. If a tail $\{x_n\}_{n\in T}$ of $\{x_n\}_{n\in U}$ converges to L,
then $\{x_n\}_{n\in U}$ converges to L.

Proof. Let $K = \max(U \smallsetminus T)$, so $K < n \in U$ implies $n \in T$.
Let M be the cost for making $\{x_n\}_{n\in T}$ match L with a tolerance ε,
so $\forall\ M \le n \in T\,,\ |x_n - L| < \varepsilon$.
Then $\max\{M, K+1\}$ is a cost for making $\{x_n\}_{n\in U}$ match L
with a tolerance ε,
since $\forall\ \max\{M, K+1\} \le n \in U\,,\ |x_n - L| < \varepsilon$. $\square$