28. Sequences

Definition. A *sequence* $\{x_n\}_{n\in U}$ or $\{x_n\}$ is a function

$$U \to \mathbb{R}; n \mapsto x_n$$

whose domain is a countably infinite subset U of $\mathbb{N}$.

Definition. Let $\{x_n\}_{n\in U}$ be a sequence.

(1) The sequence is (*monotonic*) *increasing* if

$$m < n \in U \quad \Rightarrow \quad x_m \le x_n\,.$$

(2) The sequence is (*monotonic*) *decreasing* if

$$m < n \in U \quad \Rightarrow \quad x_m \ge x_n\,.$$

(3) The sequence is *monotonic*
if it is monotonic increasing or decreasing.

Convergence of sequences. Let $\{x_n\}_{n\in U}$ be a sequence.

Definition. The sequence $\{x_n\}_{n\in U}$ *converges to a limit* L if

$$\forall\, \varepsilon > 0\,,\ \exists\, M \in \mathbb{N}\,.\ \forall\, M \le n \in U\,,\ |x_n - L| < \varepsilon\,.$$

"tolerance" ↑ ↑ "cost"

Write $\lim\limits_{n\to\infty} x_n = L\,.$

Definition. A sequence $\{x_n\}_{n\in U}$ *converges* if

$$\exists\, L \in \mathbb{R}\,.\ \lim_{n\to\infty} x_n = L\,.$$

If not, the sequence *diverges*.

Propositions.

- Convergent sequences are bounded.
- A convergent sequence has a unique limit.
- Bounded monotonic increasing $\{x_n\}_{n\in U}$ converges to $\sup\{x_n \mid n \in U\}$.
- Bounded monotonic decreasing $\{x_n\}_{n\in U}$ converges to $\inf\{x_n \mid n \in U\}$.