27. Absolute value and the Triangle Inequality

Definition. For $x \in \mathbb{R}$, the *absolute value* of x is $|x| := \sqrt{x^2}$, the distance of x from 0 on the real line.

Note

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0 \end{cases} \qquad \text{and} \qquad -|x| \leq x \leq |x|.$$

The absolute value of products. Have the equality $|x \cdot y| = |x| \cdot |y|$. In particular:

$$(*): \quad xy \leq |x| \cdot |y| \,.$$

The absolute value of sums. Only have inequality in general:

Triangle Inequality: For $x, y \in \mathbb{R}$, have $|x+y| \leq |x| + |y|$.

Proof. For $x, y \in \mathbb{R}$, inequality $(*)$ gives:

$$\begin{aligned}(x+y)^2 &= x^2 + 2xy + y^2 \\ &\leq x^2 + 2|x| \cdot |y| + y^2 = (|x| + |y|)^2 \,.\end{aligned}$$

Taking square roots yields $|x+y| \leq |x| + |y|$. $\square$

Bounded functions. Consider $f\colon D \to \mathbb{R}$.

Definition. Say f is *bounded* if its image $f(D)$ is bounded, i.e., if: $\exists\, M \in \mathbb{R} \,.\, \forall\, x \in D \,,\ |f(x)| \leq M$.

Notation: Write

$$\sup_{x \in D} f(x) := \sup f(D) \qquad \text{and} \qquad \inf_{x \in D} f(x) := \inf f(D) \,.$$