26. The Archimedean Property

Proposition. $\forall\, b \in \mathbb{R}\,,\ \forall\, 0 < t \in \mathbb{R}\,,\ \exists\, n \in \mathbb{N}\,.\ nt > b\,.$

Proof. If $b < 0$, take $n = 0$.
So now assume $b \geq 0$.
Suppose $\exists\, 0 < t \in \mathbb{R}\,.\ \forall\, n \in \mathbb{N}\,,\ nt \leq b\,.$
Thus $\forall\, n \in \mathbb{N}\,.\ n \leq b/t\,.$
Consider $m = \max\{k \in \mathbb{N} \mid k \leq b/t\}$.
Then $b/t < m + 1 \in \mathbb{N}$, a contradiction. $\square$

Corollary. Have $\inf\{1/n \mid 0 < n \in \mathbb{N}\} = 0$.

Proof. Note 0 is a lower bound for $\inf\{1/n \mid 0 < n \in \mathbb{N}\}$.
Suppose $\inf\{1/n \mid 0 < n \in \mathbb{N}\} = \varepsilon > 0$.
Set $t = \varepsilon$ and $b = 1$ in the Archimedean Property.
Thus $\exists\, 0 < n \in \mathbb{N}\,.\ n\varepsilon > 1$.
Then $\varepsilon > 1/n$, so ε is not a lower bound — contradiction! $\square$

The density of $\mathbb{Q}$ in $\mathbb{R}$. This is a major consequence of the Archimedean Property.

Proposition. $\forall\, x < y \in \mathbb{R}\,,\ \exists\, q \in \mathbb{Q}\,.\ x < q < y\,.$

Proof. Without loss of generality, assume $0 < x$.
Set $t = y - x$ and $b = 1$ in the Archimedean Property.
Thus $\exists\, 0 < n \in \mathbb{N}\,.\ n(y - x) > 1$.
Take $m = \max\{k \in \mathbb{N} \mid k < ny\}$.
Since $nx + 1 < ny$, have $nx < m < ny$.
Take $q = m/n$, so $x < m/n < y$. $\square$