25. The set of real numbers

Definition. A set is a *field* if it contains 0 and 1, and carries addition, subtraction, multiplication, and division of nonzero elements, satisfying the usual rules (associativity, commutativity, etc.).

Definition. A field is an *ordered field* if it has a total order $<$ that "plays nice with the field," so satisfies the usual properties such as

$$a < b \quad \Rightarrow a + c < b + c\,, \quad \begin{cases} ac < bc & \text{if } c > 0 \\ ac = bc & \text{if } c = 0 \\ ac > bc & \text{if } c < 0 \end{cases}, \qquad \text{etc.}$$

Definition. The set $\mathbb{R}$ of *real numbers* forms the unique ordered field, containing $\mathbb{Q}$, that has the least upper bound property.

Subsets of the real numbers. If $E \subseteq \mathbb{R}$ and $c \in \mathbb{R}$, then

$$cE := \{cx \mid x \in E\}\,, \quad -E := \{-x \mid x \in E\}\,, \quad c+E := \{c+x \mid x \in E\}\,.$$

Proposition. Suppose $\emptyset \subset E \subset \mathbb{R}$ and E is bounded below.

(1) $-E$ is bounded above.
(2) E has a g.l.b. $\inf E = -\sup(-E)$.

Maxima and minima. Consider a nonempty finite subset E of $\mathbb{R}$. Then $\sup E$ (exists and) is often called the *maximum* $\max E \in E$.
Also $\inf E$ (exists and) is often called the *minimum* $\min E \in E$.
Use same notation any time $\sup E \in E$ or $\inf E \in E$.