24. Bounds in totally ordered sets

Definition. A set S is (*totally*) *ordered* if it has a strict ordering relation $x < y$ such that the following two properties hold:

Trichotomy: For any two elements x, y of S, precisely one of the following three possibilities holds:

$$x < y \quad \text{or} \quad x = y \quad \text{or} \quad x > y\,.$$

Transitivity: For any three elements x, y, z of S, the implication

$$x < y \quad \text{and} \quad y < z \quad \text{implies} \quad x < z$$

holds.

Upper and lower bounds. Take subset E of totally ordered set S.

Definition.

- Element b of S is an *upper bound* for E if: $\forall\ x \in E\,,\ x \leq b\,.$
- Element b of S is a *lower bound* for E if: $\forall\ x \in E\,,\ b \leq x\,.$

Here, say E is respectively *bounded above* or *below* if such b exists.

Suprema and infima. Let E be a subset of a totally ordered set S.

Definition.

- An element l of S is the *supremum* or *least upper bound* (l.u.b.) $\sup E$ for E if:
  (a) l is an upper bound for E;
  (b) If b is an upper bound for E, then $l \leq b$.
- An element g of S is the *infimum* or *greatest lower bound* (g.l.b.) $\inf E$ for E if:
  (a) g is a lower bound for E;
  (b) If b is a lower bound for E, then $b \leq g$.

The least upper bound property. Let S be a totally ordered set.

Definition. Say S has the *least upper bound property* if:
whenever $\emptyset \subset E \subseteq S$ and E is bounded above, E has a l.u.b. in S.

- In $\mathbb{Q}$, $\{q \in \mathbb{Q} \mid q^2 < 2\}$ is bounded above, has no l.u.b. in $\mathbb{Q}$.
- In $\mathbb{R}$, $\{q \in \mathbb{Q} \mid q^2 < 2\}$ is bounded above, has l.u.b. $\sqrt{2}$ in $\mathbb{R}$.