23. Comparing cardinalities

For sets A, B, recall $|A| = |B|$ means there is a bijection $A \to B$.

Definition. Let A, B be sets.

(a) If there is a injective function $A \to B$,
but no surjective function $A \to B$, then $|A| < |B|$.
(b) If there is a injective function $A \to B$, then $|A| \leq |B|$.

Russell's Paradox. Recall the power set $\mathcal{P}(A)$ or 2^A of a set A.

Proposition. For any set A, have $|A| < \left|2^A\right|$.

Proof. Have injective function $A \to 2^A; a \mapsto \{a\}$.

Now suppose there is a surjective function $s\colon A \to 2^A$.
Consider $B = \{a \in A \mid a \notin s(a)\}$.
Since $s\colon A \to 2^A$ is surjective, have $b \in A$ with $s(b) = B$.

Case I: $b \in B$. Then $b \notin s(b) = B$, a contradiction.

Case II: $b \notin B$. Then $b \in s(b) = B$, a contradiction. $\square$