

22. COUNTABLE AND UNCOUNTABLE SETS

Definition. Let A be nonempty set.

- (a) If there is a surjective function $f: \mathbb{N} \rightarrow A$,
i.e., A can be written in roster notation as $A = \{a_0, a_1, a_2, \dots\}$,
then A is *countable*.
- (b) Otherwise, A is *uncountable*.
- (c) If $|\mathbb{N}| = |A|$, then A is *countably infinite*.

Theorem. A, B countable $\Rightarrow A \cup B, A \times B$ countable.

Cantor diagonalization. The set \mathbb{R} is uncountable.

Suppose $f: \mathbb{N} \rightarrow \mathbb{R}$ is surjective.

$$f(0) = n_0.\mathbf{a}_{00}a_{01}a_{02}a_{03}\dots$$

$$f(1) = n_1.a_{10}\mathbf{a}_{11}a_{12}a_{13}\dots$$

$$f(2) = n_2.a_{20}a_{21}\mathbf{a}_{22}a_{23}\dots$$

$$f(3) = n_3.a_{30}a_{31}a_{32}\mathbf{a}_{33}\dots$$

\vdots

$$\text{Choose } x = 0.\mathbf{b}_{00}\mathbf{b}_{11}\mathbf{b}_{22}\mathbf{b}_{33}\dots$$

If $\forall i \in \mathbb{N}, \mathbf{a}_{ii} \neq \mathbf{b}_{ii}$, then $x \notin f(\mathbb{N})$, a contradiction.