22. Countable and uncountable sets

Definition. Let A be a set.

(a) If there is a surjective function $f\colon \mathbb{N} \to A$,
i.e., A can be written in roster notation as $A = \{a_0, a_1, a_2, \dots\}$,
then A is *countable*. Also, $\emptyset$ is countable.
(b) Otherwise, A is *uncountable*.
(c) If $|\mathbb{N}| = |A|$, then A is *countably infinite*.

Theorem. A, B countable $\Rightarrow A \cup B$, $A \times B$ countable.

Cantor diagonalization. The set $\mathbb{R}$ is uncountable.
Suppose $f\colon \mathbb{N} \to \mathbb{R}$ is surjective.

$$\begin{aligned} f(0) &= n_0.\mathbf{a}_{00}a_{01}a_{02}a_{03}\dots \\ f(1) &= n_1.a_{10}\mathbf{a}_{11}a_{12}a_{13}\dots \\ f(2) &= n_2.a_{20}a_{21}\mathbf{a}_{22}a_{23}\dots \\ f(3) &= n_3.a_{30}a_{31}a_{32}\mathbf{a}_{33}\dots \\ &\vdots \\ \text{Choose } x &= \ 0.\mathbf{b}_{00}\mathbf{b}_{11}\mathbf{b}_{22}\mathbf{b}_{33}\dots \end{aligned}$$

If $\forall\, i \in \mathbb{N}$, $\mathbf{a}_{ii} \neq \mathbf{b}_{ii}$, then $x \notin f(\mathbb{N})$, a contradiction.