21. Bijective ≡ invertible ≡ isomorphism

Definition. On a set A, have the *identity function* $\mathrm{id}_A \colon A \to A; x \mapsto x$.

Definition. Function $f \colon A \to B; x \mapsto f(x)$ is *invertible* if there is a function $g \colon B \to A; y \mapsto g(y)$ such that $\forall\, x \in A\,,\ g(f(x)) = x$ and also $\forall\, y \in B\,,\ f(g(y)) = y$, i.e., $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

Note $g \colon B \to A$ is unique, the *inverse* $f^{-1} \colon B \to A$ of invertible f.

Definition. Function $f \colon A \to B; x \mapsto f(x)$ is *bijective* if both injective and surjective.

Bijective ≡ invertible: $\qquad f(x) = y \quad \Leftrightarrow \quad x = f^{-1}(y)$.
(existence and uniqueness of the solution $x \in A$ to $f(x) = y$ for $y \in B$)

Isomorphic sets. Say sets A and B are *isomorphic* whenever there is a bijective function $f \colon A \to B$. Then write $A \cong B$ or $|A| = |B|$ — same (possibly infinite) *cardinality.*

Definition. Function $f \colon A \to B; x \mapsto f(x)$ is a (*set*) *isomorphism* if it is bijective.

Note $|\mathbb{N}| = |A|$ means A is infinite, and can be written in roster notation as $A = \{a_0, a_1, a_2, \dots\}$.