20. Properties of functions

Consider a function $f\colon A \to B; x \mapsto f(x)$, and solving the equation

$$f(x) = y$$

for x in A, given y in B. The first definition captures "uniqueness of solutions", independently of the question of existence. The second definition captures "existence of solutions".

Definition. A function $f\colon A \to B; x \mapsto f(x)$ is *injective* (or "1-1") if the property

$$\forall\ x_1, x_2 \in A\,,\ f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

holds.

Definition. A function $f\colon A \to B; x \mapsto f(x)$ is *surjective* (or "onto") if the property

$$\forall\ y \in B\,,\ \exists\ x \in A\,.\ f(x) = y$$

holds.

Composition of functions. Given functions

$$f\colon A \to B \quad \text{and} \quad g\colon B \to C\,,$$

their *composition* is

$$g \circ f\colon A \to C; x \mapsto g\big(f(x)\big)\,.$$

Image and preimage. Consider $f\colon A \to B; x \mapsto f(x)$.

Definition. For a subset $X \subseteq A$, the subset

$$f(X) := \{f(x) \in B \mid x \in X\}$$

of B is the *image* of X.

Definition. For a subset $Y \subseteq B$, the subset

$$f^{-1}(Y) := \{x \in A \mid f(x) \in Y\}$$

of A is the *preimage* of Y.

Warning: There will be other, different usages of the notation f^{-1} later. For this reason, many authors write $f^*(Y)$ for the preimage of a subset Y of the codomain of $f\colon A \to B$, and also $f_*(X)$ for the image of a subset X of the domain.