19. Relations and functions

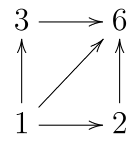
Relations on a set, or between sets. Set theory provides a language for relations, like $x \leq y$ for $x, y \in \mathbb{R}$.

Definition. For a set A, a *relation R on A* is a subset of $A \times A$.

Example. Consider $\leq$ as $\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \leq y\}$, the region in the ($x$-$y$)-plane to the left of the line $y = x$.

Notation. Write $(x, y) \in R$ as $x\,R\,y$.

Example. Consider $A = \{1, 2, 3, 6\}$ and the relation $\{(x, y) \in A \times A \mid x < y \text{ and } x \mid y\}$:



The picture is a "directed graph".

Definition. For sets A, B, a *relation R from A to B* is a subset of $A \times B$.

Functions. Functions are special kinds of relations from one set to another.

Definition. A *function* $f\colon A \to B; x \mapsto f(x)$ from *domain* set A to *codomain* set B is a relation (the *graph of the function*) from A to B such that, for each element x of the domain A, there is a unique element $f(x)$ of the codomain B such that $\big(x, f(x)\big)$ is in the relation.

Definition. The *image* [Book: "range"] of $f\colon A \to B; x \mapsto f(x)$ is

$$f(A) := \{f(x) \in B \mid x \in A\}\,.$$

Remark. The domain and codomain form an integral part of the function specification. So

$$f\colon \mathbb{R} \to [0, \infty[\,; x \mapsto x^2$$

and

$$f\colon \mathbb{R} \to \mathbb{R}; x \mapsto x^2$$

are considered as different functions.