17. Proof by induction

Proposition. $\forall\, n \in \mathbb{N}\,,\ P(n)$.

Proof. ... by induction on n.
Induction Basis: $P(0)$ holds.
Induction Step: $\forall\, n \in \mathbb{N}\,,\ P(n) \Rightarrow P(n+1)$. □

In proof by induction, $P(n)$ is the *Induction Hypothesis*.

A proof by contradiction showing proof by induction works.

Proof. Suppose $\exists\, n \in \mathbb{N}\,.\ \neg P(n)$.
Let s be the smallest natural number n for which $\neg P(n)$ holds.
Case I: $s = 0$ contradicts the Induction Basis.
Case II: If $0 < s$, then $s - 1 \in \mathbb{N}$ and $P(s-1)$ holds,
since s is the smallest natural number n for which $\neg P(n)$ holds.
By the Induction Step, $P(s-1) \Rightarrow P(s)$, so $P(s)$ holds.
This contradicts $\neg P(s)$. □

Recursive definitions. Induction proofs are often used when working with recursive definitions.

Definition. For $n \in \mathbb{N}$, the *factorial* $n!$ is defined recursively by:
Recursion Basis: $0! := 1$.
Recursion Step: $(n+1)! := (n+1) \cdot n!$.

Definition. For $n \in \mathbb{N}$, $x \in \mathbb{R}$, the *power* x^n is defined recursively by:
Recursion Basis: $x^0 := 1$.
Recursion Step: $x^{n+1} := x^n \cdot x$.