16. Binomial coefficients

Binomial Theorem. For $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$, the *binomial* $x+y$ powers as

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

with positive integral *binomial coefficients*

$$\binom{n}{k}$$

for $0 \le k \le n$.

Examples: $(x+y)^2 = x^2 + 2xy + y^2$, $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.

Computing binomial coefficients — "Pascal's Triangle".

$$\sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} y^k = (x+y)^{n+1} = (x+y)(x+y)^n$$

$$= (x+y) \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = x \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k + y \sum_{k=0}^{k=n} \binom{n}{k} x^{n-k} y^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n+1-k} y^k + \sum_{k+1=1}^{k+1=n+1} \binom{n}{(k+1)-1} x^{n-(k+1)+1} y^{k+1}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n+1-k} y^k + \sum_{k=1}^{n+1} \binom{n}{k-1} x^{n-k+1} y^k \quad \text{substituting } k+1 \to k.$$

Equating coefficients of $x^{n+1-k}y^k$ gives $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Also $\binom{n}{0} = \binom{n}{n} = 1$.

Combinatorial interpretation.

$\binom{n}{k} = \dfrac{n!}{(n-k)! \cdot k!}$. (Count k-element subsets of an n-element set.)