14. Proofs with sets

Recall $S \subseteq T$ means $x \in S \Rightarrow x \in T$.

So $\{u \in U \mid P(u)\} \subseteq \{u \in U \mid Q(u)\}$ means $\forall\, x \in U\,,\ P(x) \Rightarrow Q(x)$.

Proposition. For sets A, B, have $A \subseteq B$.

Proof. $x \in A$ implies $x \in B$. □

Proposition. For sets A, B, have $A = B$.

Proof. ($A \subseteq B$): $x \in A$ implies $x \in B$.
($B \subseteq A$): $x \in B$ implies $x \in A$. □

or (occasionally)

Proof. $x \in A \Leftrightarrow \ldots \Leftrightarrow x \in B$. □

Correspondence between set algebra and logic. Using set-builder notation:

- $\{u \in U \mid P(u)\} \cap \{u \in U \mid Q(u)\} = \{u \in U \mid P(u) \wedge Q(u)\}$
- $\{u \in U \mid P(u)\} \cup \{u \in U \mid Q(u)\} = \{u \in U \mid P(u) \vee Q(u)\}$
- $\{u \in U \mid P(u)\} \setminus \{u \in U \mid Q(u)\} = \{u \in U \mid P(u) \wedge \neg Q(u)\}$

Set algebra laws. Can be used for a "one-line proof" of set equality:

De Morgan: $\neg(A \cap B) = (\neg A) \cup (\neg B)\,,\ \neg(A \cup B) = (\neg A) \cap (\neg B)$
Idempotent: $A \cap A = A\,,\ A \cup A = A$
Distributive: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Associative: $A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C$
$A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$
Commutative: $A \cap B = B \cap A\,,\ A \cup B = B \cup A$