12. Proof by contradiction

A *contradiction* C is a false statement, like $R \wedge (\neg R)$ for statement R.

Proof by contradiction. To show statement P is true,
show that $(\neg P) \to C$ is true for some contradiction C.

Recall "[TRUE]→[FALSE]" is never true,
while "[FALSE]→[FALSE]" is true,
so if "$(\neg P)$ →[FALSE]" is true,
then $(\neg P)$ must be false, i.e., P must be true.

Proposition. Statement P holds.

Proof. $(\neg P) = P_0 \to P_1 \to \ldots \to P_{n-1} \to C$, a contradiction. □

Good news: Any contradiction C does the job.
Bad news: Starting out,
you don't know which contradiction C will do the job.

Proving implications by contradiction. Recall the logical equivalence $\neg(P \to Q) \equiv (P \wedge \neg Q)$.

Proposition. Hypothesis P implies conclusion Q.

Proof. $(P \wedge \neg Q) = P_0 \to P_1 \to \ldots \to P_{n-1} \to C$, a contradiction. □

Often takes the form:

Proposition. Hypothesis P implies conclusion Q.

Proof. $(P \wedge \neg Q) = P_0 \to P_1 \to \ldots \to P_{n-1} \to (\neg P)$, contradiction! □

Compare contrapositive:

Proposition. Hypothesis P implies conclusion Q.

Proof. $(\neg Q) = P_0 \to P_1 \to \ldots \to P_{n-1} \to (\neg P)$. □