MAXIMAL IDEALS AND ZORN'S LEMMA

1. Maximal ideals and fields

Proposition 1. *Let R be a nontrivial, commutative, unital ring. Then R is simple if and only if it is a field.*

Proof. Suppose R is simple. Suppose $0 \neq x \in R$. Then xR is a nontrivial ideal of R, so $xR = R$. Thus $\exists\ x^{-1} \in R\,.\ xx^{-1} = 1$, i.e., $x \in R^*$. Conversely, suppose R is a field and J is a nontrivial ideal of R. Consider $0 \neq x \in J$. Now $1 = xx^{-1} \in Jx^{-1} \subseteq J$. Then if $y \in R$, we have $y = 1y \in Jy \subseteq J$, so J is improper. $\square$

Definition 2. An ideal of a ring is *maximal* if it is proper, but is not properly contained in any proper ideal of the ring.

Proposition 3. *Let K be an ideal in a commutative, unital ring R. Then K is maximal if and only if R/K is a field.*

Proof. K is maximal $\Leftrightarrow$ R/K is nontrivial and simple (cf. Exercise 2) $\Leftrightarrow$ R/K is a field. $\square$

Definition 4. In the context of Proposition 3, the field R/K is called the *residue field* of the maximal ideal K.

2. Zorn's Lemma and the existence of maximal ideals

Definition 5. Let $(X, \leq)$ be a poset.

(a) A subset C of X is a *chain* or a *flag* if $c \leq d$ or $d \leq c$ for all c, d in C.
(b) An element u of X is an *upper bound* for a subset S of X if $x \leq u$ for all x in S.
(c) The poset $(X, \leq)$ is *inductive* if there is an upper bound in X for each chain in X.
(d) An element m of X is *maximal* if $m \leq x \in X$ implies $m = x$.

Remark 6. Let $\operatorname{Prop} R$ denote the set of proper ideals of a ring R. Then an ideal J of R is maximal if and only if it is a maximal element of the poset $(\operatorname{Prop} R, \subseteq)$.

The following proposition is an axiom due to Kuratowski [1]. It is equivalent to the Axiom of Choice.

Proposition 7 (Zorn's Lemma)**.** *Each inductive poset has a maximal element.*

Corollary 8. *Each proper ideal in a unital ring is contained in a maximal ideal. In particular, each nontrivial unital ring contains a maximal ideal.*

Proof. Apply Exercise 4. □

Remark 9. Exercise 5 shows that the unitality hypothesis is needed in Corollary 8.

3. Exercises

(1) Show that a nonzero ring homomorphism, whose domain is a field, is injective.

(2) Let K be an ideal of a ring R. Show that there is a bijection

$$\{J \lhd R \mid K \subseteq J\} \to \{L \mid L \lhd R/K\}; J \mapsto J/K\,.$$

[This result is sometimes called a "correspondence theorem".]

(3) (a) Show that in a commutative, unital ring, maximal ideals are prime.
  (b) Give an example of a prime ideal which is not maximal in a commutative, unital ring.

(4) Let K be an ideal of a unital ring R. Suppose that $\{J_i \mid i \in I\}$ is a chain of ideals in R, each of which contains K.
  (a) Show that K is contained in $\bigcup_{i\in I} J_i$.
  (b) Show that $\bigcup_{i\in I} J_i$ is an ideal of R.
  (c) If each ideal J_i is proper, show that $\bigcup_{i\in I} J_i$ is proper.

(5) Show that there are no maximal ideals in the zero ring given by the additive group of rational numbers.

References

[1] C. Kuratowski, *Une méthod d'élimination des nombres transfinis des raisonnements mathématiques*, Fund. Math. **3** (1922), 76–108.