THE BOOTSTRAP THEOREM

Theorem 1. *Let F be a field. Suppose that $p(X)$ is an irreducible polynomial in $F[X]$. Let J be the ideal $p(X)F[X]$ of multiples of $p(X)$ in $F[X]$. Then J is a maximal ideal in $F[X]$.*

Proof. Suppose that $f(X)$ is an element of $F[X]$ that lies outside J. Consider the ideal $K = p(X)F[X] + f(X)F[X]$ of $F[X]$. Since $F[X]$ is a principal ideal domain, there is a minimal polynomial $m(X)$ such that $K = m(X)F[X]$. Now $p(X)$, as an element of J, lies in K, so $p(X) = l(X)m(X)$ for some $l(X)$ in $F[X]$. Since $p(X)$ is irreducible, $l(X)$ or $m(X)$ is a nonzero constant. If $l(X)$ is constant, then $J = p(X)F[X] = m(X)F[X] = K$, a contradiction to $f(X)$ lying outside J. Thus $m(X)$ is constant, and $K = F[X]$. □

Corollary 2 (The Bootstrap Theorem)**.** *Under the hypotheses of the theorem, the quotient ring $F[X]/p(X)F[X]$ is a field.*