MATH 680A FALL 2016 FINAL GRADED HOMEWORK

Write clearly, on separate paper. All questions carry equal weight.
You will receive credit for your three best answers.

(1) Let H be a Hopf algebra over a field K. Suppose that j is an element of H such that

$$(*) \qquad \forall\, x \in H\,,\ xj = x^{\varepsilon} j\,.$$

(a) Show that

$$j^L \otimes xj^R = x^S j^L \otimes j^R$$

for all x in H.

(b) If H is the group algebra of a finite group G, exhibit an element j of H for which the property $(*)$ holds.

(2) Let $(H, \nabla, \Delta, \eta, \varepsilon)$ be a bialgebra over a field K.

(a) Show that $(H, \tau\nabla, \Delta, \eta, \varepsilon)$ is a bialgebra over K.

(b) If $(H, \nabla, \Delta, \eta, \varepsilon)$ is a Hopf algebra with antipode S, show that $(H, \tau\nabla, \Delta, \eta, \varepsilon)$ is a Hopf algebra if and only if S is invertible.

(3) Consider a cocommutative Hopf algebra H and a commutative Hopf algebra H' over a field K. Suppose that there is a dual pairing $\langle\ ,\ \rangle \colon H \otimes H' \to K$ between H and H'. Show that the formula

$$(\varphi \otimes h)(\theta \otimes g) = \langle h_1^S, \theta_1\rangle \theta_2\varphi \otimes h_2 g \langle h_3, \theta_3\rangle\,,$$

for $h, g \in H$ and $\varphi, \theta \in H'$, defines an associative product on $H' \otimes H$.

(4) Show that the quantum double $D(G)$ of a finite group satisfies the quasitriangular axiom $(\Delta \otimes 1_{D(G)})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}$.